BOOK 2, Raison D'etre; The PDE of the Statistically Accurate Option Premium

The first four pages of the *Raison D'etre* **is a first principles derivation of** $rV - V_t = \frac{1}{2}\sigma^2 S^2 V_{SS} + (\mu + \frac{1}{2}\sigma^2) S V_S$, the partial differential equation (for the relevant boundary conditions) of a statistically unbiased option premium V at time t for an option to buy, or an option to sell, a proportional Brownian motion market asset S at time T for a fixed and predetermined price k. The premium V, as the solution to this equation $rV - V_t = \frac{1}{2}\sigma^2 S^2 V_{SS} + (\mu + \frac{1}{2}\sigma^2) S V_S$, is the fair premium for someone who wants the option exposure, i.e., access to the right but not the obligation to buy or sell the market asset S for k at time T. It is reminded that a person whom wants the option exposure and the payoff profile of the option must not take an offsetting position in the underlying asset.

A proportional Brownian motion has diffusion equation

$$\frac{dS_t}{dt} = \left(\mu + \tfrac{1}{2}\sigma^2\right)S_t + \sigma S_t \frac{dB_t}{dt} \qquad \frac{dS(t)}{dt} = \left(\mu + \tfrac{1}{2}\sigma^2\right)S(t) + \sigma S(t)\frac{dB(t)}{dt}$$

where B_t is standard Brownian motion, $B_t \sim N(0,t)$.

(1) The Transition Distribution of the Market Asset

Taking the logarithm of the diffusion variable S_t for the underlying additive Brownian motion $\log S_t$; by a Taylor series, the total derivative of the transformed variable $\log S_t$ is

$$\begin{aligned}\frac{d\log S_t}{dt} &= \frac{\partial \log S_t}{\partial t}\frac{dt}{dt} + \frac{\partial \log S_t}{\partial S_t}\frac{dS_t}{dt} + \tfrac{1}{2}\frac{\partial^2 \log S_t}{\partial S_t^2}\frac{(dS_t)^2}{dt} + \ldots \\ &= \frac{\partial \log S_t}{\partial t} + \frac{\partial \log S_t}{\partial S_t}\frac{dS_t}{dt} + \tfrac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 \log S_t}{\partial S_t^2} + \frac{o(dt)}{dt}\end{aligned}$$

since $(dS_t)^2 = \sigma^2 S_t^2 dt + o(dt)$ and $(dS_t)^n = o(dt)$ for $n > 2$.

From **(1)** $\frac{\partial \log S_t}{\partial t} = 0$ **(2)** $\frac{\partial \log S_t}{\partial S_t} = \frac{1}{S_t}$ **(3)** $\frac{\partial^2 \log S_t}{\partial S_t^2} = -\frac{1}{S_t^2}$ and $\frac{o(dt)}{dt} \to 0$

$$\begin{aligned}\frac{d\log S_t}{dt} &= \frac{1}{S_t}\left(\left(\mu + \tfrac{1}{2}\sigma^2\right)S_t + \sigma S_t \frac{dB_t}{dt}\right) + \tfrac{1}{2}\sigma^2 S_t^2\left(-\frac{1}{S_t^2}\right) \\ &= \mu + \sigma\frac{dB_t}{dt}\end{aligned}$$

which is an ordinary Brownian motion in the variable $\log S_t$. Conditioning at time t and integrating over the transition interval from t to T for the forward transition distribution

$$\int_t^T \frac{d\log S_y}{dy}dy = \int_t^T \left(\mu + \sigma\frac{dB_y}{dy}\right)dy$$

which results in

$$\begin{aligned}&\left[\log S_y\right]_t^T = \left[\mu y + \sigma B_y\right]_t^T \\ &\log S_T - \log S_t = \mu(T-t) + \sigma(B_T - B_t) \\ &\log S_T = \log S_t + \mu(T-t) + \sigma(B_T - B_t) \\ &S_T = \exp\left(\log S_t + \mu(T-t) + \sigma(B_T - B_t)\right)\end{aligned}$$

From distribution theory, conditional on $S(t) = S_t$ at time t is a constant

$$\begin{aligned}&B_T - B_t \sim N(0, T-t) \\ &\sigma(B_T - B_t) \sim N\left(0, \sigma^2(T-t)\right) \\ &\log S_t + \mu(T-t) + \sigma(B_T - B_t) \sim N\left(\log S_t + \mu(T-t), \sigma^2(T-t)\right)\end{aligned}$$

and exponentiating for a log-normal

$$\exp\left(\log S_t + \mu(T-t) + \sigma(B_T - B_t)\right) \sim \log N\left(\log S_t + \mu(T-t), \sigma^2(T-t)\right)$$

Hence, the transition distribution is log-normal

$$S_T \mid S_t \sim \log N\left(\log S_t + \mu(T-t), \sigma^2(T-t)\right)$$

The density function of this distribution, that of the market asset S to deliver at time T given that the spot price is S_t at time t, is

$$f_{S(T)|S(t)}(x \mid S_t) = \frac{1}{x\sigma\sqrt{T-t}\sqrt{2\pi}} \exp\left(-\frac{[\log x - \log S_t - \mu(T-t)]^2}{2\sigma^2(T-t)}\right)$$

(2) The Backward Equation of the Transition Density of the Market Asset

Again, taking logarithms for the additive process $\log S$ which has a normal transition distribution

$$\log S_T \mid \log S_t \sim N\left(\log S_t + \mu(T-t), \sigma^2(T-t)\right)$$

The transition moment generating function (MGF) of the additive process $\log S$ is then

$$M_{\log S_T|\log S_t}(\theta) = \exp\left(\theta \log S_t + \mu(T-t)\theta + \tfrac{1}{2}\sigma^2(T-t)\theta^2\right)$$

Differentiating the MGF partially with respect to the backward time variable t

$$\frac{\partial}{\partial t} M_{\log S_T|\log S_t}(\theta) = \left(-\mu\theta - \tfrac{1}{2}\sigma^2\theta^2\right) M_{\log S_T|\log S_t}(\theta)$$

Differentiating the MGF partially with respect to the logarithm of the backward diffusion variable $\log S_t$

$$\frac{\partial}{\partial \log S_t} M_{\log S_T|\log S_t}(\theta) = \theta M_{\log S_T|\log S_t}(\theta)$$

and again

$$\frac{\partial^2}{\partial \log S_t^2} M_{\log S_T|\log S_t}(\theta) = \theta^2 M_{\log S_T|\log S_t}(\theta)$$

Substituting in the expressions for $\theta M_{\log S_T|\log S_t}(\theta)$ and $\theta^2 M_{\log S_T|\log S_t}(\theta)$ into the time derivative

$$\frac{\partial}{\partial t} M_{\log S_T|\log S_t}(\theta) = \left(-\mu \frac{\partial}{\partial \log S_t} - \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial \log S_t^2}\right) M_{\log S_T|\log S_t}(\theta)$$

and inverting the MGF for the transition density function

$$-\frac{\partial}{\partial t} f_{\log S_T|\log S_t}(\log x \mid \log S_t) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial \log S_t^2} f_{\log S_T|\log S_t}(\log x \mid \log S_t) + \mu \frac{\partial}{\partial \log S_t} f_{\log S_T|\log S_t}(\log x \mid \log S_t)$$

which is the backward equation of the underlying additive Brownian motion $\log S$ where

$$f_{\log S_T|\log S_t}(\log x \mid \log S_t) = \frac{1}{\sigma\sqrt{T-t}\sqrt{2\pi}} \exp\left(-\frac{[\log x - \log S_t - \mu(T-t)]^2}{2\sigma^2(T-t)}\right)$$

Transforming the partial derivatives and the diffusion variable

(1) $\dfrac{\partial}{\partial \log S_t} = S_t \dfrac{\partial}{\partial S_t}$ **(2)** $\dfrac{\partial^2}{\partial \log S_t^2} = \dfrac{\partial}{\partial \log S_t}\left(\dfrac{\partial}{\partial \log S_t}\right) = S_t \dfrac{\partial}{\partial S_t}\left(S_t \dfrac{\partial}{\partial S_t}\right) = S_t^2 \dfrac{\partial^2}{\partial S_t^2} + S_t \dfrac{\partial}{\partial S_t}$

and noting the Jacobian of the transformation in the forward diffusion variable is $|J| = x$

$$f_{\log S_T|\log S_t}(\log x \mid \log S_t) = x\, f_{S_T|S_t}(x \mid S_t)$$

for

$$-\frac{\partial}{\partial t} x\, f_{S_T|S_t}(x \mid S_t) = \tfrac{1}{2}\sigma^2 \left(S_t^2 \frac{\partial^2}{\partial S_t^2} + S_t \frac{\partial}{\partial S_t}\right) x\, f_{S_T|S_t}(x \mid S_t) + \mu S_t \frac{\partial}{\partial S_t} x\, f_{S_T|S_t}(x \mid S_t)$$

which, upon simplification and cancelation of the Jacobian (since it is the backward equation so the Jacobian, which is a function of the forward variable, can be moved outside the partial derivatives), becomes

$$-\frac{\partial}{\partial t} f_{S_T|S_t}(x \mid S_t) = \tfrac{1}{2}\sigma^2 S_t^2 \frac{\partial^2}{\partial S_t^2} f_{S_T|S_t}(x \mid S_t) + \left(\mu + \tfrac{1}{2}\sigma^2\right) S_t \frac{\partial}{\partial S_t} f_{S_T|S_t}(x \mid S_t)$$

This is the backward equation of the market asset to deliver *S*. **In this page and the next (pages 3 and 4) the dependent variable *f* of this equation is transformed to that of the statistically accurate option premium *V***

$$rV(T,k \mid S_t,t) - \frac{\partial}{\partial t} V(T,k \mid S_t,t) = \tfrac{1}{2}\sigma^2 S_t^2 \frac{\partial^2}{\partial S_t^2} V(T,k \mid S_t,t) + \left(\mu + \tfrac{1}{2}\sigma^2\right) S_t \frac{\partial}{\partial S_t} V(T,k \mid S_t,t)$$

This equation $rV - V_t = \frac{1}{2}\sigma^2 S^2 V_{SS} + (\mu + \frac{1}{2}\sigma^2) S V_S$ is the purpose and reason, and hence the Raison D'etre, for BOOK 2 of this PhD Dissertation[1].

(3) The Backward Equation of the Statistically Accurate Call Option Premium

An option is the right to buy or sell the market asset *S* for the predetermined price *k* at time *T*. The option premium at time *t* is the present value of the first limited moment above or below the exercise price *k* depending on whether the option is a call or a put. For a right to call the asset *S* to buy for *k* at time *T* in $T - t$ periods time, the option premium is

$$\begin{aligned} V(T,k \mid S_t,t) &= e^{-r(T-t)} E\left(\max(S_T - k) \mid S_t\right) \\ &= e^{-r(T-t)} \int_k^\infty (x-k) f_{S(T)|S(t)}(x \mid S_t) dx \end{aligned}$$

which evaluates upon integration (omitted) to the statistically accurate call option premium

$$V(T,k \mid S_t,t) = S_t e^{(\mu + \frac{1}{2}\sigma^2 - r)(T-t)} \overline{F}_U\left(u_k - \sigma\sqrt{T-t}\right) - k e^{-r(T-t)} \overline{F}_U\left(u_k\right)$$

where *T* and *t* are the forward and backward time variables respectively, S_t is the backward diffusion variable and spot price of the underlying market asset at time *t*, $\overline{F}_U(\cdot)$ is the complementary cumulative distribution function of a standard unit normal $U \sim N(0,1)$

$$\overline{F}_U(u) = 1 - F_U(u) = \int_u^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

and u_k is a normal pivot of the exercise value *k* log-standardized by the transition cumulants of the market asset *S*

$$u_k = \frac{\log k - \log S_t - \mu(T-t)}{\sigma\sqrt{T-t}}$$

From the backward equation of the market asset to deliver

$$-\frac{\partial}{\partial t} f_{S_T|S_t}(x \mid S_t) = \tfrac{1}{2}\sigma^2 S_t^2 \frac{\partial^2}{\partial S_t^2} f_{S_T|S_t}(x \mid S_t) + \left(\mu + \tfrac{1}{2}\sigma^2\right) S_t \frac{\partial}{\partial S_t} f_{S_T|S_t}(x \mid S_t)$$

multiplying by the intrinsic value, which is a function of the forward diffusion variable *x*, and integrating over the range of the forward diffusion variable corresponding to the region of option moneyness, which for a call, is anything greater than the exercise price *k* for which the asset can be called to buy at time *T*

$$-\int_k^\infty (x-k) \frac{\partial}{\partial t} f_{S_T|S_t}(x \mid S_t) dx = \int_k^\infty (x-k) \left(\tfrac{1}{2}\sigma^2 S_t^2 \frac{\partial^2}{\partial S_t^2} f_{S_T|S_t}(x \mid S_t) + \left(\mu + \tfrac{1}{2}\sigma^2\right) S_t \frac{\partial}{\partial S_t} f_{S_T|S_t}(x \mid S_t) \right) dx$$

Taking the integrals inside the partial derivatives which are derivatives in the backward variables

$$-\frac{\partial}{\partial t} \int_k^\infty (x-k) f_{S_T|S_t}(x \mid S_t) dx = \left(\tfrac{1}{2}\sigma^2 S_t^2 \frac{\partial^2}{\partial S_t^2} + \left(\mu + \tfrac{1}{2}\sigma^2\right) S_t \frac{\partial}{\partial S_t} \right) \int_k^\infty (x-k) f_{S_T|S_t}(x \mid S_t) dx$$

and discounting both sides over the forward transition of $T - t$ periods from *T* to *t* at the cash rate *r* with the discount factor $e^{-r(T-t)}$

[1] as it became clear to me that the debased option premium is not a sensible premium and that the Black-Scholes-Merton equation $rV^{debased} - V_t^{debased} = \frac{1}{2}\sigma^2 Y^2 V_{YY}^{debased} + r Y V_Y^{debased}$ is not a sensible equation for derivatives.

$$-e^{-r(T-t)}\frac{\partial}{\partial t}\int_k^\infty (x-k)\, f_{S_T|S_t}(x\,|\,S_t)dx = e^{-r(T-t)}\left(\tfrac{1}{2}\sigma^2 S_t^2\frac{\partial^2}{\partial S_t^2}+\left(\mu+\tfrac{1}{2}\sigma^2\right)S_t\frac{\partial}{\partial S_t}\right)\int_k^\infty (x-k)\, f_{S_T|S_{t_0}} f_{S_T|S_t}(x\,|\,S_t)dx$$

Taking the discount factor $e^{-r(T-t)}$ inside the partial derivatives; that in the backward time variable adds a source (by derivative of parts) as a diminishing sink as option maturity at time *T* nears for increases in *t* and the effect of the discounting diminishes

$$\left(r-\frac{\partial}{\partial t}\right)e^{-r(T-t)}\int_k^\infty (x-k)\, f_{S_T|S_t}(x\,|\,S_t)dx = \left(\tfrac{1}{2}\sigma^2 S_t^2\frac{\partial^2}{\partial S_t^2}+\left(\mu+\tfrac{1}{2}\sigma^2\right)S_t\frac{\partial}{\partial S_t}\right)e^{-r(T-t)}\int_k^\infty (x-k)\, f_{S_T|S_t}(x\,|\,S_t)dx$$

With $V(T,k\,|\,S_t,t)=e^{-r(T-t)}\int_k^\infty (k-x)\, f_{S(T)|S(t)}(x\,|\,S_t)dx$ this is the PDE of the statistically unbiased and accurate call option premium[2]

$$\left(r-\frac{\partial}{\partial t}\right)V(T,k\,|\,S_t,t) = \left(\tfrac{1}{2}\sigma^2 S_t^2\frac{\partial^2}{\partial S_t^2}+\left(\mu+\tfrac{1}{2}\sigma^2\right)S_t\frac{\partial}{\partial S_t}\right)V(T,k\,|\,S_t,t)$$

i.e.,

$$rV(T,k\,|\,S_t,t)-\frac{\partial}{\partial t}V(T,k\,|\,S_t,t) = \tfrac{1}{2}\sigma^2 S_t^2\frac{\partial^2}{\partial S_t^2}V(T,k\,|\,S_t,t)+\left(\mu+\tfrac{1}{2}\sigma^2\right)S_t\frac{\partial}{\partial S_t}V(T,k\,|\,S_t,t)$$

subject to the boundary condition of the value of the option if the asset to deliver *S* is called to buy

$$V(T,k\,|\,S_T,T)=\max\left(S_T-k,0\right)$$

The PDE of the accurate call option premium is $rV - V_t = \tfrac{1}{2}\sigma^2 S^2 V_{SS} + (\mu+\tfrac{1}{2}\sigma^2)\, S\, V_S$ written in shorthand for the option premium *V* as the dependent variable and the partial derivatives with respect to *t* and *S* as subscripts. The derivation is similar for a put option (omitted).

If one alternatively demeans the asset *S* upon which the option is written by removing all excess return net of cash so that it is a debased asset with full market risk, then does all this maths again for demeaned asset *Y* [the oxymoron risky bank account]

$$\log Y = \log S-(\mu+\tfrac{1}{2}\sigma^2-r)(T-t) \qquad\qquad Y_T\,|\,Y_t \sim \log N\left(\log Y_t+(r-\tfrac{1}{2}\sigma^2)(T-t),\sigma^2(T-t)\right)$$

then one arrives at the Black-Scholes-Merton PDE $rV^{debased} - V_t^{debased} = \tfrac{1}{2}\sigma^2 Y^2 V_{YY}^{debased} + r\, Y\, V_Y^{debased}$ and the Black and Scholes put and call prices $V^{debased}$ (1973). The PDE and premiums are wrong for the market asset to deliver *S*; it is the PDE for an option on the wrong asset and the independent variable must be *Y* rather than *S* unless it is being subsidized by trading *S* in which case the replicating asset attempts to subsidize the mispricing in the option premium. This subsidization has nothing to do with being risk free, but rather only flatlining the option. The argument about an option portfolio being risk free is flawed and only results in a debased option portfolio and a mispriced option. This is what markets have been indicating since 1973 with things like the volatility skew and put-call parity for the Black-Scholes prices not holding for market prices; this is a result of market prices (which recognize return) systematically deviating from the debased prices of Black and Scholes (as noted in Black and Scholes 1973 for call options). The market knows better. Risk-neutral valuation is the wrong mind set for efficient markets and a technically adept industry; it stands in the way of a bright future for options (efficient markets for risk based upon correctly valued options), the profession[3] (and some wonderful and exciting mathematics of BOOK 3 of my PhD based on probability theory and calculus being applied to options), those parts of the economy that depend directly or indirectly on the financial markets, and the full recognition to the economic value of derivatives as derived from the spot markets.

The accurate PDE just derived $rV - V_t = \tfrac{1}{2}\sigma^2 S^2 V_{SS} + (\mu+\tfrac{1}{2}\sigma^2)\, S\, V_S$ is the PDE for the correct price of an option derivative. The important point about this equation is that it has the correct infinitesimal mean which is the return on the underlying market asset. This is not the cash rate. **It is only the cash rate if the premium has been debased. How this occurs will now be shown in the next two pages (pages 5 and 6).**

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[2] written in a (de)convolution format as $$\frac{\partial V_t}{\partial t} = rV_t-\left(\mu+\tfrac{1}{2}\sigma^2\right)S_t\frac{\partial V_t}{\partial S_t}-\tfrac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V_t}{\partial S_t^2}$$

[3] whom will do a much better job of preventing derivative disasters than risk neutral valuation has.

A. Diffusion Equation of the Statistically Accurate Option Premium

This is an intermediate step to deducing the return on replicating the option with an offsetting delta position in the underlying asset. If one reasons what the portfolio earns net of the delta update, then this and the next step (**A** and **B**) can be skipped (this page, **page 5**). Do so at your own peril, though, as risk-neutral valuation has (and my thesis has demonstrated both in discrete time and continuous time).

Expanding the option premium in a Taylor series

$$\frac{dV_t}{dt} = \frac{\partial V_t}{\partial t}\frac{dt}{dt} + \frac{\partial V_t}{\partial S_t}\frac{dS_t}{dt} + \tfrac{1}{2}\frac{\partial^2 V_t}{\partial S_t^2}\frac{(dS_t)^2}{dt}$$
$$= \frac{\partial V_t}{\partial t} + \frac{\partial V_t}{\partial S_t}\frac{dS_t}{dt} + \tfrac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V_t}{\partial S_t^2}$$

The total derivative is forward moving through time, over which the option moves closer to expiry and the backward variables of the option premium move toward the forward variables located at the option expiry at time *T*. The dynamics in the option premium is that of time decay and the partial time derivative of the option premium that describes this is that of the backward equation, that of the diffusion side of the backward equation with a source for time decay in the discount factor

partial derivative

$$\frac{\partial V_t}{\partial t} = rV_t - \left(\mu + \tfrac{1}{2}\sigma^2\right)S_t\frac{\partial V_t}{\partial S_t} - \tfrac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V_t}{\partial S_t^2} \qquad V_t = V(T,k\,|\,S_t,t)$$

From my thesis, substituting this equation for the partial time derivative $\partial V_t/\partial t$ of the option into the total derivative (which will be back out again on the next page)

$$\frac{dV_t}{dt} = \left(rV_t - (\mu + \tfrac{1}{2}\sigma^2)S_t\frac{\partial V_t}{\partial S_t} - \tfrac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V_t}{\partial S_t^2}\right) + \frac{\partial V_t}{\partial S_t}\frac{dS_t}{dt} + \tfrac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V_t}{\partial S_t^2}$$

which is the diffusion equation of the option premium [original to this thesis]

total derivative

$$\frac{dV_t}{dt} = rV_t - (\mu + \tfrac{1}{2}\sigma^2)S_t\frac{\partial V_t}{\partial S_t} + \frac{\partial V_t}{\partial S_t}\frac{dS_t}{dt} \qquad \text{where } \frac{\partial V_t}{\partial S_t} = h_t \text{ is the } \textbf{instantaneous option delta}.$$

B. Return on Replicating the Statistically Accurate Option Premium

Taking an offsetting position in the underlying asset S_t (to flatline the option) with delta h_t units of the market asset to replicate the instantaneous option delta and kill the option V_t. Conditional on S_t the portfolio Π_t is then

$$\Pi_t = h_t S_t - V_t$$

It is now proved that the replicating portfolio Π need not earn a flat rate of return at cash and only does so if it has been debased, either with a debased asset or a debased option. Otherwise, it earns the market rate of return on the market asset *S* and incurs the cash rate on the short option *V* net of the delta update due. Noting that $dS_t d\Delta_t$ is not $o(dt)$

$$\frac{d\Pi_t}{dt} = (S_t + dS_t)\frac{dh_t}{dt} + h_t\frac{dS_t}{dt} - \frac{dV_t}{dt}$$

ignoring the update $(S_t + dS_t)\frac{dh_t}{dt}$ the total portfolio change for a fixed delta is

$$\frac{d\Pi_t}{dt} = h_t\frac{dS_t}{dt} - \frac{dV_t}{dt}$$

noting that rigor requires the update and o(*dt*) with limits. The results are the same and I have left these out purely to keep the important arguments succinct. Substituting in the total derivative of the option

$$\frac{d\Pi_t}{dt} = h_t\frac{dS_t}{dt} - \left(rV_t - (\mu + \tfrac{1}{2}\sigma^2)h_t S_t + h_t\frac{dS_t}{dt}\right)$$
$$= (\mu + \tfrac{1}{2}\sigma^2)h_t S_t - rV_t$$

which shows that the portfolio earns the market rate of return on the market asset *S* and incurs the cash rate on the short option *V*. In other words, the portfolio does not earn the cash rate ($d\Pi_t \neq r\Pi_t dt$) unless and only unless it has been debased; either a debased asset or a debased premium. Whenever the portfolio has not been debased and the premium is

accurate for the market asset to deliver, the replicating portfolio earns the market return on the market asset and incurs the cash rate on the short option $d\Pi_t = \left((\mu + \frac{1}{2}\sigma^2)h_t S_t - rV_t\right)dt$.

The market return is the instantaneous economic value of the derivative derived from taking up the pecuniary interest in the economic asset of the deferred spot transaction of the derivative. A debased return flat at *r* implies that the derivative has no economic value and is consequently a redundant security.

C. Reversing out again the PDE of the Option Premium

If one debases the portfolio by constraining its dynamics fixed flat at the cash rate, $d\Pi_t = r\Pi_t dt$, then you debase the portfolio of the benefits of trading the market asset and participating in the economic activity of the markets. This results in either a debased asset or a debased premium, in either case, a premium that is debased of its mean net of cash; if the asset has not been debased, the premium debases it by sinking the return of the asset from the mispricing in the debased premium. If the PDE of the option premium is reversed out, this PDE is that of the debased premium, the Black-Scholes-Merton equation. If the PDE is reversed out of the correct returns on the portfolio, which were just proved to be $d\Pi_t = \left((\mu + \frac{1}{2}\sigma^2)h_t S_t - rV_t\right)dt$, then the PDE of the correct and statistically accurate premium is obtained. From **(1#)** the PDE of the accurate premium is backed out from the portfolio that has not been debased of the benefits of markets, and from (**2#)** the PDE of the debased premium is backed out from the debased portfolio

(1#) $d\Pi_t = \left(\boldsymbol{(\mu + \frac{1}{2}\sigma^2)h_t S_t - rV_t}\right)\boldsymbol{dt}$ (portfolio not debased)

(2#) $d\Pi_t = \boldsymbol{r\Pi_t dt}$ (portfolio debased)

where $\boldsymbol{r\Pi_t dt = \left(rh_t S_t - rV_t\right)dt}$. From $\frac{d\Pi_t}{dt} = (S_t + dS_t)\frac{dh_t}{dt} + h_t\frac{dS_t}{dt} - \frac{dV_t}{dt}$, again expanding the total derivative of the option in a Taylor series

$$\frac{d\Pi_t}{dt} = (S_t + dS_t)\frac{dh_t}{dt} + h_t\frac{dS_t}{dt} - \left(\frac{\partial V_t}{\partial t}\frac{dt}{dt} + \frac{\partial V_t}{\partial S_t}\frac{dS_t}{dt} + \frac{1}{2}\frac{\partial^2 V_t}{\partial S_t^2}\frac{(dS_t)^2}{dt}\right)$$

$$= (S_t + dS_t)\frac{dh_t}{dt} - \frac{\partial V_t}{\partial t} - \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V_t}{\partial S_t^2}$$

after canceling the deltas, the replicating delta with the instantaneous option delta. Note that this is not risk free because the replicating delta must be instantaneously updated in *dt* with $(S_t + dS_t)dh_t/dt$ if it is to be kept matched with the instantaneous option delta. The delta update $(S_t + dS_t)dh_t/dt$ has both mean and variance; this then maintains the instantaneously delta neutral portfolio [which is impossible in practice]. Holding this constant for the differential transition interval from *t* to $t + dt$ (ignoring $(S_t + dS_t)dh_t/dt$), this is just another expression for the total portfolio change net of the update

(#) $\boldsymbol{\frac{d\Pi_t}{dt} = -\frac{\partial V_t}{\partial t} - \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V_t}{\partial S_t^2}}$

This only differs from that of the portfolio return components in that it has not had the PDE of the option substituted into it for

$$\frac{\partial V_t}{\partial t}$$

and instead, it is just reversed out again when this is equated as an identity ($d\Pi_t \equiv d\Pi_t$) to the portfolio return components for which the PDE has been substituted for (exactly similarly for the binomial portfolio $\Delta\Pi_j \equiv \Delta\Pi_j$);

(1) equating **(#)** to the correct portfolio returns **(1#)**

$$\boldsymbol{-\frac{\partial V_t}{\partial t} - \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V_t}{\partial S_t^2} = (\mu + \frac{1}{2}\sigma^2)h_t S_t - rV_t}$$

$$-\frac{\partial V_t}{\partial t} = \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V_t}{\partial S_t^2} + (\mu + \frac{1}{2}\sigma^2)S_t\frac{\partial V_t}{\partial S_t} - rV_t$$

correct PDE (PDE of statistically accurate option)

$$rV - V_t = \tfrac{1}{2}\sigma^2 S^2 V_{SS} + (\mu + \tfrac{1}{2}\sigma^2)\,S\,V_S$$

(2) equating **(#)** to the debased portfolio returns **(2#)**

$$\boldsymbol{-\frac{\partial V_t}{\partial t} - \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V_t}{\partial S_t^2} = rh_t S_t - rV_t}$$

$$-\frac{\partial V_t}{\partial t} = \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V_t}{\partial S_t^2} + rS_t\frac{\partial V_t}{\partial S_t} - rV_t$$

BSM PDE (PDE of debased option)

$$rV^{debased} - V_t^{debased} = \tfrac{1}{2}\sigma^2 S^2 V_{SS}^{debased} + r\,S\,V_S^{debased}$$

which just reverses out the PDE which was substituted into the total derivative of the option premium and consequently into the total derivative of the portfolio; one PDE correct, the other PDE wrong as a result of debasing the portfolio of market return with $d\Pi_t = r\Pi_t dt$. **This is how an option premium gets debased**. It is shown quite clearly in the binomial context. Theoretically, debasing the portfolio with $d\Pi_t = r\Pi_t dt$ introduces a slack variable Δ into the mean which debases the mean and portfolio return to the cash rate r; $\mu' = \mu - \Delta$

$$\mu' + \tfrac{1}{2}\sigma^2 = r \qquad d\Pi_t = r\Pi_t dt$$
$$(\mu - \Delta) + \tfrac{1}{2}\sigma^2 = r \qquad = \left((\mu' + \tfrac{1}{2}\sigma^2)h_t S_t - rV_t\right)dt$$
$$\Delta = \mu + \tfrac{1}{2}\sigma^2 - r \qquad = \left((\mu - \Delta + \tfrac{1}{2}\sigma^2)h_t S_t - rV_t\right)dt$$

i.e., the economic return on the portfolio $\mu + \frac{1}{2}\sigma^2$ is debased to $\mu' + \frac{1}{2}\sigma^2 = \mu + \frac{1}{2}\sigma^2 - \Delta = r$. The slack variable Δ is then backed out again[4] with implied volatility and results in the volatility skew $\sigma(\Delta, k)$ over k **and term to maturity**. How does one get an accurate price from a debased price? One does so, as a better alternative to skewing volatility, by re-meaning the backward variable by boosting the spot price S_t to

$$S_t e^{(\mu + \frac{1}{2}\sigma^2 - r)(T-t)}$$

and using this as the spot price in the debased formula, for either a put or a call. This is one of several methods of converting the debased PDE $rV^{debased} - V_t^{debased} = \frac{1}{2}\sigma^2 Y^2 V_{YY}^{debased} + r\,Y\,V_Y^{debased}$ of Black, Scholes, and Merton to the statistically accurate and correct PDE $rV - V_t = \frac{1}{2}\sigma^2 S^2 V_{SS} + (\mu + \frac{1}{2}\sigma^2)\,S\,V_S$ derived by BOOK 2 of this thesis (and again in **pages 1 to 4** in this *Raison D'etre*) and the resulting solution for the accurate premium. An empirically observed value of this quantity (the adjusted spot) can be obtained from market observed put-call parity

$$S_t e^{(\mu + \frac{1}{2}\sigma^2 - r)(T-t)} = \hat{C} - \hat{P} + ke^{-r(T-t)}$$

of the statistically accurate formula where $\hat{C} - \hat{P} + ke^{-r(T-t)}$ is observed from market prices $\hat{C}$ and $\hat{P}$. Interestingly, **matching market put-call parity** will mean that one will then never have problems with put-call parity violations! Note that this implies a market rate of return, inferred from the derivative markets, of

$$\mu + \tfrac{1}{2}\sigma^2 - r = \frac{1}{T-t}\log\frac{\hat{C} - \hat{P} + ke^{-r(T-t)}}{S_t} \qquad \text{or} \qquad \mu + \tfrac{1}{2}\sigma^2 = r + \frac{1}{T-t}\log\frac{\hat{C} - \hat{P} + ke^{-r(T-t)}}{S_t}$$

net of cash (where S_t is the spot price). I am also advocating implied return-volatility regions from statistically accurate formulae for V from $rV - V_t = \frac{1}{2}\sigma^2 S^2 V_{SS} + (\mu + \frac{1}{2}\sigma^2)\,S\,V_S$, those with both mean and variance, equated to market prices. This gives another indicator of return on the region somewhere around market volatility calculated directly from market data on the underlying asset. My thesis also works out formulae for confidence intervals and regions for option prices with respect to returns and volatility. This thesis advocating strongly that **return be fully recognized and appreciated in the derivative markets** as extensions to the spot markets as deferred transactions of those markets, **and consequently, that the economic value of derivatives be fully recognized rather than being considered as debased and redundant securities**. Under risk-neutral valuation, if it worked which it doesn't, all options are dead options; why? simply to subsidize them by killing their payoff profiles. The connotation? the PDE of the statistically accurate option premium $rV - V_t = \frac{1}{2}\sigma^2 S^2 V_{SS} + (\mu + \frac{1}{2}\sigma^2)\,S\,V_S$ is the sensible PDE for derivatives, not that which has been debased of market return and debased of the economic benefits of a security derived from the spot markets, i.e., not the PDE for redundant option securities $rV^{debased} - V_t^{debased} = \frac{1}{2}\sigma^2 Y^2 V_{YY}^{debased} + r\,Y\,V_Y^{debased}$.

(4) References

Lloyd-Jones, A., (2011) *Stochastic Calculus and Option Valuation: A Perspective from Mathematical Statistics*, Ph. D. Dissertation. Discipline of Finance, the University of Sydney.

Black, F. and Scholes, M. J., (1973) The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, Vol. 81, No. 3 (May-Jun., 1973), pp. 637-659.

[4] a bad habit of risk-neutral valuation; backing out variables and equations after making bad judgement in equating various quantities, which are then trashed with slack.

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PART I: LOG-BINOMIAL ASSETS, OPTIONS, AND REPLICATING PORTFOLIOS

Purpose and Introduction

The purpose and objective of this book is to explain the statistical bias in the risk-neutral (return-neutered) formulae for debased options and make an argument for accurate premiums with a Revised Philosophy for Option Markets to Best Estimates of Unbiased Premia which it is believed is **a consistent conceptual framework for derivative markets** that recognizes market return and the economic value of derivatives as deferred transactions of the spot markets. This conceptual framework is opposed to the view that derivatives are redundant securities. Derivatives become redundant after being debased and then flatlined. Risk-neutral valuation is synonymous with the debasement of options and is consistent with the socialist system of communism whereby benefits of markets are barred. This thesis is arguing against the debasement of options and other derivatives, and presenting the alternative *best estimates of accurate premiums*. This argument is based upon a sound mathematical argument.

I would not have written this book if risk-neutral valuation did not bar accurate prices; this thesis would have consisted only of the content of BOOK 1 and BOOK 3 as a calculus to model market variables and obtain accurate premiums which, as it stands, are not allowed. Debased premiums need to be relegated to history. If you do not want to spend your time reading the next 80 or so pages, then simply look at the debased option premiums and ask yourself "where is the mean"? Whenever an option premium does not have the correct mean for the asset for which it is priced, then it is statistically biased and an inaccurate measure of value. The arguments specifically for debased option portfolios and premiums, and why they are not a sufficient reason to systematically misprice options, start on page 41. The focus before this is on **(1)** statistically accurate premiums and **(2)** replicating portfolios for statistically accurate premiums, i.e., replicating portfolios that have not been debased.

Definitions and Notation for Variables and Parameters

This section introduces the variables and statistical foundation of options on a market asset S that varies as a multiplicative binomial random walk, the option on this market asset and the fair and accurate premium for the option, and a replicating portfolio Π *consisting of an option and offsetting asset to replicate and offset the delta and binomial basis of the option to flatline the payoff profile of the option.* ***The return components of*** Π *are then derived and it is proved irrefutably for the accurate premium, by several alternative methods, that the offsetting market asset earns the market return and the short option incurs the cash rate (combinatorially and then in a discrete calculus setting). The equation of the NPV of cashflows for the portfolio is obtained by integrating the return components and this result shows that, for the portfolio of the accurate premium trading the market asset to replicate, the option is flatlined in return for the returns earned on trading the market asset, for which the derivative writer takes the up pecuniary interest in the economic asset for the option holder and the deferred transaction of the derivative. Debased option portfolios uses these returns earned on flatlining an option to justify systematically mispricing options at debased prices.* ***It should be clear to common sense that there is no point to flatlining options unless either one can earn something [by trading it for market returns] or one doesn't like options. If it is wanted that no one should benefit from the option markets and that derivatives should remain debased, then the option markets should be shut down.***

The definitions and notation for the (i) variables and (ii) parameters of this book are: time j (backward time variable), expiry n (forward time variable), exercise price k, upstep ratio $u = 1+\Delta_u$, downstep ratio $d = 1-\Delta_d$, (where Δ_u, Δ_d are the effective rates of return and discount for upsteps and downsteps respectively), upstep probability p, cash rate r (effective for the step length), discount factor v ($v = (1 + r)^{-1}$), binomial market asset S_j (the spot price and backward walk variable), option premium $f_{j,n}$, delta of option premium $h_{j,n}$, replicating portfolio $\Pi_{j,n}$, minimum (maximum) number of upsteps to expire inthemoney $a(S_j)$ where this is the smallest (largest) integer greater (smaller) than $\log(k/S_j d^{n-j})/\log(u/d)$ for a call (put) option; additionally for a debased portfolio and premium, Y_j is the demeaned binomial asset and the distribution of this asset is referred to by the literature as "the risk-neutral measure" which is a misnomer. A debased premium is referred to by the literature as the "risk-neutral price" which is also a misnomer. Primed p or u (p' or u') are debasement parameters of Π, and Δ is the slack variable for p' or u' which debases the option portfolio and premium. The statistical bias of a debased premium is the difference between the debased premium and the statistically accurate premium for the market asset S to deliver which is called to buy or put to sell if the option expires inthemoney. For discrete calculus, the symbol δ denotes the partial difference operator whilst Δ denotes the ordinary (total) difference operator. The context should make it clear whether Δ is a difference or slack.

The transition **probability mass function** (PMF) of the step generator and number of upsteps for a transition from time j to time n is $p_X = p_{X_n|X_j}(x|x_0)$ and the PMF of the proportional (exponential) random walk of the market asset for that transition interval is $p_{S_n|S_j}(y|s_0)$. Words in square brackets or braces are additional nested commentary for more depth.

The Proportional Binomial Random Walk and Binomial Market Asset S

The binomial model and delta-neutral replicating portfolio relate to a hypothetical market asset S which is a proportional binomial random walk. This is an unmoderated exponential diffusion variable with a discrete two-point basis function [i.e., a sequence of coin tosses]. At each step of the proportional binomial random walk for S, the walk moves upward by a multiplicative factor of u or down by a multiplicative factor of d, with probabilities determined by the Bernoulli basis variables $X_j \sim B(1, p)$ in a fixed sum of the step generator and walk basis function. The return on S is then an ordinary additive walk, i.e., the logarithm of S changes with discrete additive upsteps of $\log u$ and downsteps of $\log d$. Considering the forward variable of the proportional walk at the end of n steps; this is the market asset S and forward value of the underlying asset to the option for a forward transition of n steps. The forward transition of the walk variable is a geometric function of **(1)** the steps of the walk, **(2)** the spot value of the asset at time zero S_0, and **(3)** the binomial basis function. Given S_0, the n step ahead market asset S is

$$S_n = S_0 u^X d^{n-X}$$

where $u > 1$ and $0 < d < 1$, and where $X = X_1 + X_2 + ... + X_n \sim B(n, p)$ is the sum of n Bernoulli basis variables generating the two-valued risk. These are zero or one indicator random variables indicating whether the j^{th} step

was a step upwards or a step downwards. X is consequently the number of n forward steps which are upwards. The remaining $n - X$ steps are therefore downsteps. The PMF of the total number of upsteps (without the backward variables of X) is

$$p_X(x) = {}^nC_x p^x (1-p)^{n-x} \qquad x = 0,1,2,...,n$$

so the PMF of the walk is (from the COMPENDIUM)

$$p_{S_n|S_0}(y \mid S_0) = p_X\left(\log(y/S_0 d^n) \Big/ \log(u/d)\right)$$
$$= {}^nC_{\log(y/S_0 d^n)/\log(u/d)}\, p^{\log(y/S_0 d^n)/\log(u/d)} q^{\log(S_0 u^n / y)/\log(u/d)}$$

for $\quad y = S_0 d^n,\ S_0 u^1 d^{n-1}, S_0 u^2 d^{n-2},\ S_0 u^3 d^{n-3} ...,\ S_0 u^{n-2} d^2,\ S_0 u^{n-1} d^1,\ S_0 u^n$

The mean of the forward transition is

$$E(S_n \mid S_0) = \sum_{x=0}^{n} S_0 u^x d^{n-x}\, {}^nC_x p^x q^{n-x} = S_0 \sum_{x=0}^{n} {}^nC_x (pu)^x (qd)^{n-x} = S_0 (pu+qd)^n$$

which is n periods of compound returns at an effective rate of market return $puv + qdv - 1$ net of cash. The variance of the n step forward transition is

$$Var(S_n \mid S_0) = S_0^2 (pu^2 + qd^2)^n - S_0^2 (pu+qd)^{2n}$$
$$= S_0^2 \left\{ \left(pq(u-d)^2 + (pu+qd)^2 \right)^n - (pu+qd)^{2n} \right\}$$

which is analogous to log-normal variance for an exponential Brownian motion. For a single step, the market asset changes from S_0 to S_1 which is either $S_0 u$ with probability p or $S_0 d$ with probability $q = 1 - p$

$$S_1 = \begin{cases} S_0 u & \text{with probability} \quad p \\ S_0 d & \text{with probability} \quad q \end{cases}$$

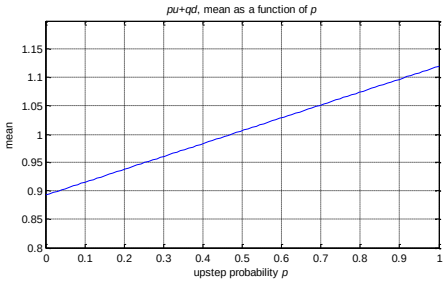
The average size of the market asset at the end of the first step given S_0 is

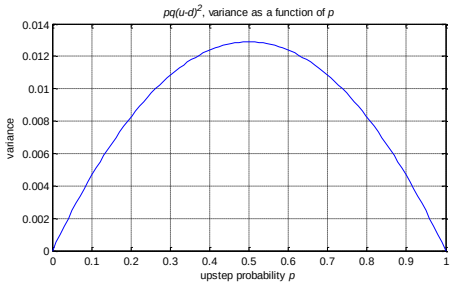
$$E(S_1 \mid S_0) = p(S_0 u) + q(S_0 d)$$
$$= (pu+qd) S_0$$

which is an average effective compound return of $E(S_1 - S_0 \mid S_0)/S_0 = pu + qd$ on the market asset S per step and is a market return of $puv + qdv - 1$ net of cash. The variance of a single step in the market asset S is

$$Var(S_1 \mid S_0) = S_0^{\,2} pq(u-d)^2$$

so the variance of the return is $Var(S_1 - S_0 \mid S_0)/S_0^{\,2} = pq(u-d)^2$ for which the important point is that by changing the probability of an upstep p, the variance of return is changed in the opposite direction to the change in the mean return, i.e., for positively returning binomial assets ($p > 0.50$), an increase in p increases the mean return whilst at the same time decreasing the variance of return so that higher return is associated with lower risk.





This is not consistent with, and instead contrary to, the generally accepted notion that there is a positive relationship between risk and return. The variance of S is at a quadratic maximum for $p = 0.50$ and decreases on either side of this maxima for any variation in p.

The Statistically Unbiased Option Premia for the Binomial Market Asset S

Letting f be the option premium for the hypothetical proportional binomial market asset S which returns u if it moves up and d if it moves down over any step, for an average effective return of $pu + qd - 1$ per binomial step. $f_{0,n}(k \mid S_0)$ is the option premium at time 0 for the option to buy or sell the underlying asset for k in n periods time given that the spot price of the underlying asset is S_0. The notation f_0 and $f_{0,n}$ are shorthand for the initial premium.

Subscripts on the market asset S and the option premium f denote time when it is a number (or j) and to the direction of the steps leading up to the current values of the variables when it is a letter, u or d. In a single steps time, the option premium becomes

$$f_{1,n}(k \mid S_1) = \begin{cases} f_{1,n}(k \mid S_0 u) \\ f_{1,n}(k \mid S_0 d) \end{cases} = \begin{cases} f_u & \text{if the asset moves to } S_u = S_0 u \\ f_d & \text{if the asset moves to } S_d = S_0 d \end{cases}$$

which is one of two possible values as dependent upon the two possible values of the underlying market asset S. f_u and f_d are shorthand for the two alternative premiums conditional on whether the first step was up or down. Similarly,

$$f_{2,n}(k \mid S_2) = \begin{cases} f_{2,n}(k \mid S_0 u^2) \\ f_{2,n}(k \mid S_0 ud) \\ f_{2,n}(k \mid S_0 d^2) \end{cases} = \begin{cases} f_{uu} & \text{if the asset moves to } S_{uu} = S_0 u^2 \\ f_{ud} & \text{if the asset moves to } S_{ud} = S_0 ud \\ f_{dd} & \text{if the asset moves to } S_{dd} = S_0 d^2 \end{cases}$$

and so on, where subscripts on f, for example f_{ud}, commute. The call option is the right but not the obligation to call the market asset to buy for the fixed predetermined price k in n steps time whilst the put option is the right but not the obligation to put the market asset to sell for the fixed predetermined price k in n steps time. The payoff of the option is the intrinsic value of the option at option expiry at the end of the n^{th} step. For a call option, this is then

$$\max\left(S_n - k,\ 0 \mid S_0\right) = \max\left(S_0 u^X d^{n-X} - k, 0\right) \qquad \text{for} \quad X \sim B(n, p)$$

and for a put option

$$\max\left(k - S_n,\ 0 \mid S_0\right) = \max\left(k - S_0 u^X d^{n-X}, 0\right) \qquad \text{for} \quad X \sim B(n, p)$$

The call option is a deferred long position in the market asset and the put option is a deferred short position in the market asset, if the price is favourable to the option at maturity. These are therefore deferred spot transactions with the right [option] to turn down an unfavourable price. The value of **(1)** the deferral of the spot transaction is the value of the derivative, and **(2)** the right to turn down the deferred transaction is the value of the option over that of a future. A put option is the option market equivalent to a short trade in the spot markets (as a call option is the option market equivalent to a long trade in the spot markets). The fair and accurate value of the call or put option is then the expected present value of the intrinsic value of the option on the market asset to deliver S at the end of n steps, i.e., for a forward transition of n steps. Hence, the discounted expected value of these functions are the call and put option premiums for an option that expires in n steps time. With the statistical formula for the mean of a transformation,

$$E\left(g(X)\right) = \sum_{x \in X} g(x) \times p_X(x)$$

where the transformation is the payoff at maturity, and discounting for the premium

$$f = v^n E\left(g(X)\right) = v^n \sum_{x \in X} g(x) \times p_X(x)$$

For the call option premium $g(X) = \max(S_0 u^X d^{n-X} - k, 0)$ so

$$\begin{aligned} f_{0,n} &= v^n E\left(\max(S_0 u^X d^{n-X} - k, 0)\right) \\ &= v^n \sum_{x=0}^{n} \max\left(S_0 u^x d^{n-x} - k, 0\right) p_X(x) \\ &= v^n \sum_{x=0}^{n} \max\left(S_0 u^x d^{n-x} - k, 0\right) {}^nC_x p^x q^{n-x} \end{aligned}$$

and for the put option premium $g(X) = \max(k - S_0 u^X d^{n-X}, 0)$ so

$$\begin{aligned} f_{0,n} &= v^n E\left(\max(k - S_0 u^X d^{n-X}, 0)\right) \\ &= v^n \sum_{x=0}^{n} \max\left(k - S_0 u^x d^{n-x}, 0\right) p_X(x) \\ &= v^n \sum_{x=0}^{n} \max\left(k - S_0 u^x d^{n-x}, 0\right) {}^nC_x p^x q^{n-x} \end{aligned}$$

The max function for the excess function of the call option expiry is non-zero whenever $S_0 u^x d^{n-x} \geq k$. In terms of the variable of summation, this solves for $x \geq \log(k / S_0 d^n) / \log(u / d)$ so that the terms of summation for the call option premium are non-zero over this part of the sample space of the forward transition in *S*. For a put option, the max function for the shortfall function is non-zero whenever $S_0 u^x d^{n-x} \leq k$. The inequality in terms of the variable of summation for the put option premium is then $x \leq \log(k / S_0 d^n) / \log(u / d)$. Letting *a* be **(1)** the **smallest integer greater than** $\log(k / S_0 d^n) / \log(u / d)$ for a **call** or **(2)** the **greatest integer smaller than** $\log(k / S_0 d^n) / \log(u / d)$ for a **put**. This is the nominal log-standardized value of the exercise price of the option and is **(1)** the **minimum number of upsteps** for a **call** to expire inthemoney and **(2)** the **maximum number of upsteps** for a **put** to expire inthemoney. A non-zero value of *a* is the condition that the option has non-redundant optionality and consequently is an option proper (rather than a pseudo option). The premium for a call option expiring in *n* steps time is

$$\begin{aligned} f_{0,n} &= v^n \sum_{x=a}^{n} \left(S_0 u^x d^{n-x} - k\right) {}^nC_x p^x q^{n-x} k \\ &= S_0 v^n \sum_{x=a}^{n} {}^nC_x (pu)^x (qd)^{n-x} - k v^n \sum_{x=a}^{n} {}^nC_x p^x q^{n-x} \\ &= S_0 (vpu + vqd)^n \sum_{x=a}^{n} {}^nC_x \left(\frac{pu}{pu+qd}\right)^x \left(\frac{qd}{pu+qd}\right)^{n-x} - k v^n \sum_{x=a}^{n} {}^nC_x (vp)^x (vq)^{n-x} \\ &= S_0 (vpu + vqd)^n \bar{F}_{B(n,p')}(a) - k v^n \bar{F}_{B(n,p)}(a) \end{aligned}$$

where $\bar{F}_{B(n,p')}(\cdot)$ and $\bar{F}_{B(n,p')}(\cdot)$ are the **complementary cumulative mass functions** (complementary CMFs) of binomial variables with parameters **(i)** *n* and $p' = pu / (pu + qd)$ and **(ii)** *n* and *p* respectively. The formula for the put option premium is

$$\begin{aligned} f_{0,n} &= v^n \sum_{x=0}^{a} \left(k - S_0 u^x d^{n-x}\right) {}^nC_x p^x q^{n-x} \\ &= k v^n \sum_{x=0}^{a} {}^nC_x p^x q^{n-x} - S_0 v^n \sum_{x=0}^{a} {}^nC_x (pu)^x (qd)^{n-x} \\ &= k v^n \sum_{x=0}^{a} {}^nC_x (vp)^x (vq)^{n-x} - S_0 (vpu + vqd)^n \sum_{x=0}^{a} {}^nC_x \left(\frac{pu}{pu+qd}\right)^x \left(\frac{qd}{pu+qd}\right)^{n-x} \\ &= k v^n F_{B(n,p)}(a) - S_0 (vpu + vqd)^n F_{B(n,p')}(a) \end{aligned}$$

where $F_{B(n,p')}(\cdot)$ and $F_{B(n,p')}(\cdot)$ are the **cumulative mass functions** (CMFs) of binomial variables with parameters **(i)** *n* and *p* and **(ii)** *n* and $p' = pu / (pu + qd)$ respectively. The factor $vpu + vqd$ is the excess return over the cash rate *r* so that $(vpu + vqd)^n$ is *n* periods of compounded market return net of cash [but probably not inflation]. The summations can be evaluated by a binomial table lookup or software function call for F or $\bar{F}$ or by successive recursive discounted expectations from the expiry values on a spreadsheet with the recursion formula for the premium directly from the backwards deconvolution recurrence

$$f_{j,n}(k \mid S_0) = vp\, f_{j+1,n}(k \mid S_j u) + vq\, f_{j+1,n}(k \mid S_j d)$$

subject to the terminus of the lattice at the forward variables with *n*+1 binomial nodes $S_n = S_0 u^x d^{n-x} \quad x = 0,1,\ldots,n$

$$f_{j,n}(k \mid S_0) = \begin{cases} \max\left(S_n - k, 0 \mid S_n\right) \text{ for a call} \\ \max\left(k - S_n, 0 \mid S_n\right) \text{ for a put} \end{cases}$$

This is solved for the initial value by finishing the backwards recursion with

$f_{0,n}(k \mid S_0) = vp\, f_{1,n}(k \mid S_0 u) + vq\, f_{1,n}(k \mid S_0 d)$

i.e., $f_0 = vp\, f_u + vq\, f_d$, which can be unfolded to the boundary conditions for the required premium. This recurrence is derived and proved in a proceeding topic (page 20 and the four pages starting on page 29**Error! Bookmark not defined.**) in order to prove that the replicating portfolio for this premium, that which is accurate for the market asset to deliver, earns the market return on the market asset and incurs the cash rate on the option, and then further, that the mispricing in the debased premium is subsidized by trading returns earned by replicating and flatlining the option. Related recursions involving sums of binomial probabilities, distribution and complementary binomial distributions, and other expected values can be used to calculate higher order raw moments and from these, central moments such as variance, skewness, and kurtosis. If cumulative binomial probabilities are used, there are two table lookups; one for the CMF of the asset with probability parameter *p* and one for the CMF of the adjusted distribution for the excess returns from the return factors with probability parameter $p' = pu / (pu + qd)$.

Solving the recurrence for the option premium accurate for the market asset to deliver

The backwards recurrence equation for the backward dynamics of the option premium $f_0 = vpf_u + vqf_d$ is solved for the required accurate premium at the initial backward variables by successively unfolding the steps to the boundary condition; the backward recurrence for one forward step in the backward variables is

$f_u = vpf_{uu} + vqf_{du}$ if the spot increases and $f_d = vpf_{ud} + vqf_{dd}$ if the spot decreases

and again, the backwards recurrence for a total of two forward steps in the backward variables

$f_{uu} = vpf_{uuu} + vqf_{uud} \qquad f_{ud} = vpf_{udu} + vqf_{udd} \qquad f_{dd} = vpf_{ddu} + vqf_{ddd}$

and so on. Unfolding the backwards recursion for the option premium up to the forward variables at step *n*

$$\begin{aligned} f_0 &= vpf_u + vqf_d \\ &= vp\left(vpf_{uu} + vqf_{ud}\right) + vq\left(vpf_{ud} + vqf_{dd}\right) \\ &= v^2 p^2 f_{uu} + 2v^2 pqf_{ud} + v^2 q^2 f_{dd} \\ &= v^2 p^2 \left(pf_{uuu} + qf_{uud}\right) + 2v^2 pq\left(pf_{udu} + qf_{udd}\right) + v^2 q^2 \left(pf_{ddu} + qf_{ddd}\right) \\ &= v^3 p^3 f_{uuu} + 3v^3 p^2 qf_{uud} + 3v^3 pq^2 f_{udd} + v^3 q^3 f_{ddd} \\ &\ldots \\ &= v^n \sum_{x=0}^{n} {}^nC_x\, p^x q^{n-x} f_{u^x d^{n-x}} \end{aligned}$$

For which the boundary conditions specify the intrinsic value of the deferred transaction. The option premium at the boundary coincides with the expiry value. For the call option, this is $f_{u^x d^{n-x}} = \max\left(S_0 u^x d^{n-x} - k, 0\right)$ so that

$$\begin{aligned} f_0 &= v^n \sum_{x=0}^{n} {}^nC_x\, p^x q^{n-x} f_{u^x d^{n-x}} \\ &= v^n \sum_{x=0}^{n} {}^nC_x\, p^x q^{n-x} \max\left(S_0 u^x d^{n-x} - k, 0\right) \\ &= (puv + qdv)^n \bar{F}_{B(n,p')}(a) - kv^n \bar{F}_{B(n,p)}(a) \end{aligned}$$

and the option premium at the boundary for the put option is $f_{u^x d^{n-x}} = \max\left(k - S_0 u^x d^{n-x}, 0\right)$ so that

$$\begin{aligned} f_0 &= v^n \sum_{x=0}^{n} {}^nC_x\, p^x q^{n-x} f_{u^x d^{n-x}} \\ &= v^n \sum_{x=0}^{n} {}^nC_x\, p^x q^{n-x} \max\left(k - S_0 u^x d^{n-x}, 0\right) \\ &= kv^n F_{B(n,p)}(a) - (puv + qdv)^n F_{B(n,p')}(a) \end{aligned}$$

as required and as per expectation of the step generator *X* transformed to the option expiry $v^n E\left(g(X) \mid S_0\right)$.

Market Put-Call Parity and the Futures Premium

Market put-call parity of the binomial market asset is

$c_{0,n} - p_{0,n} = S_0 (puv + qdv)^n - kv^n$

where c is the binomial call option premium and p is the binomial put option premium. This, market put-call parity, is also the [undebased] premium for a long future

$$F_{0,n} = c_{0,n} - p_{0,n} = S_0(puv + qdv)^n - kv^n$$

for a derivatives market for which prices are not debased, and instead recognize the return of the underlying spot market from which the value of derivatives derive. Given that market prices for derivatives are not debased [despite misleading theory], as observed for debased put-call parity not holding for option prices, then current market practice for futures for which there is no premium ($F = 0$) implies that the exercise price k [the futures price] is set so that the futures premium is zero

$$F_{0,n} = c_{0,n} - p_{0,n} = S_0(puv + qdv)^n - kv^n = 0$$

This then implies that all futures are written so that they expire for the future expected value of the underlying asset $k = E(S_n \mid S_0)$ [with a market assessment for future return based on the interaction of supply and demand of deferred spot transactions]. For a binomial market asset this is

$$kv^n = S_0(puv + qdv)^n$$
$$k = S_0(pu + qd)^n$$

It is suggested that this restriction, for zero premium for futures prices [by letting k vary into parity with $E(S_n \mid S_0)$] be lifted so that k can be set by negotiation, although something nearthemoney S_0 , for which a premium can be exchanged upon writing the future [although not ruling out setting the exercise price at its future value $k = E(S_n \mid S_0)$; i.e., not changing market practice for futures other than allowing more flexibility and sophistication in setting prices]. The short position receives the futures premium F from the long position for a future transaction at k. For negative drift, the short position receives a negative futures premium, and hence pays the long side to enter into the position. The workings of the futures markets then more closely follow that of the option markets and are then more consistent for *A Consistent Framework for Derivatives* and deferred transactions of the spot markets. In the case whereby the futures market does not have the depth or volume, or is not liquid enough to have various values of the exercise price k [or any other reason], then current practice as it stands should be recognized as adjusting the exercise price for the future value of the underlying spot market. This then gives another method of remeaning debased option prices and utilizing the futures markets as a crystal ball for returns and sentiment of the spot markets, similarly for the option markets, as will be discussed.

A. THE REPLICATING PORTFOLIO FOR THE ACCURATE OPTION PREMIUM

The Option Replicating Portfolio and Option Delta for the Binomial Market Asset S

A delta-neutral option replicating portfolio Π is established by taking an offsetting position in the underlying market asset S, in proportion to a quantity h, to match and offset the binomial basis function of the option f derived from the asset S upon which the value of the option depends

$$\Pi_{j,n} = h_{j,n}S_j - f_{j,n}$$

It is proved[1] (rigorously from first principles with the eight pages from page 29, original to this thesis) that the total derivative of the option premium is

$$\frac{\Delta f_{j,n}}{\Delta j} = rf_{j,n} - (pu + qd - 1)S_j h_{j,n} + h_{j,n}\frac{\Delta S_j}{\Delta j} \qquad \text{where} \quad \frac{\Delta S_j}{\Delta j} = \begin{cases} S_j(u-1) \text{ with probability } p \\ S_j(d-1) \text{ with probability } q \end{cases}$$

so that an offsetting position in delta $h_{j,n}$ units of the market asset is sufficient to kill the binomial variation ΔS_j of the option net of the residual variation for the delta changes [note that debasing the premium also kills the signal]. Introducing delta h for this thesis is the purpose of this topic. For no other walk variable is delta suf-

[1] this is a main result and perhaps I am introducing it a bit early, but if this is substituted into the total portfolio change then the return components result; the market return on the asset and the cash rate on the option [giving one of the main acts of this exposition away a little early].

ficient. Taking an offsetting position in the underlying asset and replicating the option [to nullify it] with the offsetting asset then kills the option's delta and nets out the binomial basis function. The remaining portfolio variation is born by the delta rebalances with trading (buys and sells) in S to keep the replicating delta matched with that of the option and keep ΔS_j out of the net portfolio cashflows. This amounts to n buys and sells in S for an n-step forward transition to option expiry. This is sourced directly from equity so that the balance sheet always depends on binomial risk ΔS_j for any option with non-redundant optionality ($a > 0$) if the offsetting position in the asset is to keep the delta matched to that of the option and maintain the replication. Otherwise the replication is broken and void.

The total change $\Delta\Pi_j$ in the replicating portfolio is composed of a change in the position in the market asset $\Delta h_j S_j$ and a change in the value of the option premium Δf_j .

$$\frac{\Delta\Pi_{j,n}}{\Delta j} = \frac{\Delta(h_{j,n}S_j)}{\Delta j} - \frac{\Delta f_{j,n}}{\Delta j}$$

The total change in the delta position of the asset $\Delta(h_{j,n}S_j)$ contained in the delta-neutral portfolio is composed of two items, as a result of one form of the discrete product rule, and both of which in the product, S_{j+1} and Δh_j being dependent upon the stochastic term ΔS_j ; **(1)** a change in the value of the market asset $h_j\Delta S_j$ for a fixed delta and **(2)** a change in the delta for a forward value of the market asset $S_{j+1}\Delta h_j$;

$$\begin{aligned}\frac{\Delta\Pi_{j,n}}{\Delta j} &= h_{j,n}\frac{\Delta S_j}{\Delta j} + S_{j+1}\frac{\Delta h_{j,n}}{\Delta j} - \frac{\Delta f_{j,n}}{\Delta j} \\ &= h_{j,n}\frac{\Delta S_j}{\Delta j} + (S_j + \Delta S_j)\frac{\Delta h_{j,n}}{\Delta j} - \frac{\Delta f_{j,n}}{\Delta j}\end{aligned}$$

The former $h_{j,n}\Delta S_j$ changes within the portfolio by the asset change and the latter $S_{j+1}\Delta h_{j,n}$ must be born out of equity if the replicating asset is to keep the delta killed and the replication is to remain unbroken to option maturity. The delta rebalance $S_{j+1}\Delta h_j$ is an essential part of the portfolio changes and composes a risk to equity, rather than an asset risk, since it is financed directly from the rebalancer's wallet [cash at bank]. In the case of a call option for example, if the market asset S decreases in value, then the delta decreases meaning that the devalued asset must be shorted resulting in a loss making trade. **If any one of the *n* rebalances are missed, then the replication is broken and the missed delta update accumulates into portfolio gamma risk, and higher order terms of stochastic variation accumulate with each further missed rebalance. The delta-neutral arguments are then void**. It is proved that the return for a portfolio which trades the market asset S to replicate and offset the statistically unbiased premium with the delta of the unbiased premium earns a market rate of return on the delta quantity of the market asset, which is $pu + qd - 1$, and incurs the cash rate r on the short option (as valued by its premium). This portfolio only earns the cash rate r if (and only if) either **(1)** the risky market asset S returns only the cash rate (but has 100% of the market risk of S) or **(2)** the premium has been debased which sinks all of the excess return on the market asset S by incurring a loss on the short option greater, not only than the cash rate, but greater than the return on the excess returning market asset, i.e., in excess of both r and $pu + qd - 1$. This additional loss incurred on the debased option is in proportion to delta and the market return which sinks returns earned on trading the market asset. This amounts to the market returns earned on trading the market asset subsidizing the mispricing in the debased premium. This has been the justification to debase and systematically misprice options.

If the underlying asset moves upwards from time zero to time one, then the value of the portfolio becomes $\Pi_{1,n} = h_{1,n}S_1 - f_{1,n}$. Just before the delta update is paid for, out of cash at bank and consequently equity, the value of the portfolio is $\Pi_{1,n} = h_{0,n}S_1 - f_{1,n}$ for an unchanged delta which is constant within the portfolio until the mandatory rebalance is paid to keep the offsetting asset matched to the option delta. Since the hypothetical security is binomial, this is either $\Pi_u = h_0S_u - f_u$ if the spot price has moved upwards in value or $\Pi_d = h_0S_d - f_d$ if it has moved downwards in value

$$\begin{aligned}\Pi_{1,n} &= h_{0,n}S_1 - f_{1,n} \\ &= \begin{cases} \Pi_u = h_0S_u - f_u & \text{if an upward movement occurred in } S \\ \Pi_d = h_0S_d - f_d & \text{if a downward movement occurred in } S \end{cases}\end{aligned}$$

By investing (paying out cash) for the offsetting position in the underlying asset to establish the portfolio and offset the option (and essentially kill the option), the portfolio value is the same either way between the binomial lattice nodes and between the mandatory delta updates that occur at the nodes since delta is sufficient for two-valued risk [but not any other form of basis risk]. Hence, $\Pi_u = \Pi_d$

$$h_0 S_u - f_u = h_0 S_d - f_d$$

which solves for the initial delta,

$$h_0 = \frac{f_u - f_d}{S_u - S_d}$$

Risk neutral valuation assumes further that the portfolio is not only the same whether it is a step up or a step down but also that the portfolio return is constrained flat at the cash rate $\Pi_0 = v\Pi_u = v\Pi_d$. This is wrong if the underlying market asset does not grow, on average, at the cash rate and the portfolio has not been debased with a mispriced premium, in which case, the return is neither fixed nor flat but instead an interaction between the return on the asset and the cash rate on the option. This is proved shortly by both combinatorial methods and discrete calculus. Constraining the portfolio return only has the effect of inadvertently debasing the option portfolio and demeaning the market asset (castrating it of return), and consequently backing out a debased option premium that is void of market return and mispriced for the asset to deliver *S*. Note that if the portfolio meets the requirements for full immunization (either $k = 0$ or $a = 0$ for redundant optionality), it would achieve full immunization in which case it would be possible to take profits from the deltas and **allocate earned premium for the service provided by the derivative**; *k* or *a* are not ever zero in actual option markets, so an option portfolio never achieves full immunization and the position is consequently ALWAYS subject to risk due to statistically varying deltas. For options proper, a deterministic portfolio return, net of the rebalance, is only possible for two valued risk since then (and only then) is killing the option's delta sufficient; it is not possible for any other diffusion variable, not even three valued risk, let alone infinite-valued risk of markets since killing delta is not sufficient for any variable other than a walk with a two-valued binomial basis variable [for example, not sufficient for trinomial walks, tetranomial walks, or any other form of walk with a discrete or continuous basis function].

The delta is the partial divided difference and the discrete partial derivative of the option premium with respect to the underlying asset. This is the ratio of the (vertical) distance between the two alternative nodes on the lattice of the option premium to the (vertical) difference between the two alternative nodes on the lattice of the underlying walk. The delta for the first step is therefore set at the start of the lattice at the first originating node of the lattice, by reference to the two point distribution of values at the subsequent binomial nodes at the end of the two vertices. The initial delta just before the first step as conditioned on S_0 can be written in discrete differential form as

$$h_{0,n} = \frac{\delta f_{1,n}}{\delta S_1}$$

where $f_{1,n}$ and S_1 refer to the values of the premium and asset at one of the two respective possible nodes following the first step; given that movements upwards are proportional with *u* and movements downward with *d*

$$h_{0,n} = \frac{f_{1,n}(k \mid S_u) - f_{1,n}(k \mid S_d)}{S_u - S_d} = \frac{f_{1,n}(k \mid S_0 u) - f_{1,n}(k \mid S_0 d)}{S_0 u - S_0 d} = \frac{f_u - f_d}{S_0(u - d)}$$

as before. The rebalances for delta $S_1 \Delta h_{0,n}$ is due at the start of the step for payment at the end of the step, since it is not known at the start conditional on S_0 and delta is matched to that of the option for only one discrete instantaneous moment, i.e., only momentarily for a single step and therefore must be maintained for a total of *n* updates and trades. In continuous time, the instantaneous delta is neutralized for only one infinitesimal moment and therefore it must be sourced continuously [which is impossible]. Successive deltas for the start of any particular step are similarly calculated as the partial difference of the option premium with respect to the underlying asset at the end of that particular step, and the updates then maintain the momentary delta from buys and sells out of equity. If the spot price of the market asset appreciates with an upwards step to $S_0 u$ over the period from time zero to time one, then noting that $k = S_0 u \times u^{a-1} d^{n-(a-1)}$, (so that the exercise price is on the lattice so as not to unnecessarily complicate matters and instead keep the maths concise) then the call option premium appreciates to

$$
\begin{aligned}
f_u &= v^{n-1}\sum_{x=0}^{n-1} \max\left(S_0 u \times u^x d^{n-1-x} - k, 0\right) {}^{n-1}C_x p^x q^{n-1-x} \\
&= v^{n-1}\sum_{x=a-1}^{n-1} \left(S_0 u^{x+1} d^{n-1-x} - k\right) {}^{n-1}C_x p^x q^{n-1-x} \\
&= v^{n-1}(S_0 u^a d^{n-a} - k) {}^{n-1}C_{a-1} p^{a-1} q^{n-a} + v^{n-1}\sum_{x=a}^{n-1} \left(S_0 u^{x+1} d^{n-1-x} - k\right) {}^{n-1}C_x p^x q^{n-1-x} \\
&= v^{n-1}(k-k) {}^{n-1}C_{a-1} p^{a-1} q^{n-a} + v^{n-1}\sum_{x=a}^{n-1} \left(S_0 u^{x+1} d^{n-1-x} - k\right) {}^{n-1}C_x p^x q^{n-1-x} \\
&= v^{n-1}\sum_{x=a}^{n-1} \left(S_0 u^{x+1} d^{(n-1)-x)} - k\right) {}^{n-1}C_x p^x q^{(n-1)-x}
\end{aligned}
$$

and so the value of the option premium following an upward step is $f_{1,n}(k \mid S_1) = f_{1,n}(k \mid S_0 u)$ and there is one less step in forward transition to maturity, and the backward variable of the walk is proportionately larger by a factor of u net of time decay. If the spot price of the market asset depreciates with a downwards step to $S_0 d$ between time zero and time one, then the call option premium becomes

$$
\begin{aligned}
f_d &= v^{n-1}\sum_{x=0}^{n-1} \max\left(S_0 d \times u^x d^{n-1-x} - k, 0\right) {}^{n-1}C_x p^x q^{n-1-x} \\
&= v^{n-1}\sum_{x=a}^{n-1} \left(S_0 u^x d^{n-x} - k\right) {}^{n-1}C_x p^x q^{n-(x+1)}
\end{aligned}
$$

so similarly for a downward step, $f_{1,n}(k \mid S_1) = f_{1,n}(k \mid S_0 d)$, where there is one less step in forward transition to maturity, and the backward variable of the walk is proportionately smaller by a factor of d (for $d < 1$). The difference between the two alternative values of the option premium is then the partial difference and vertical distance between two nodes on the lattice of the option premium, $\delta f_{1,n}$ with respect to S_1

$$
\begin{aligned}
f_u - f_d &= v^{n-1}\sum_{x=a}^{n-1} \left(S_0 u^{x+1} d^{n-(x+1)} - k\right) {}^{n-1}C_x p^x q^{n-(x+1)} - v^{n-1}\sum_{x=a}^{n-1} \left(S_0 u^x d^{n-x} - k\right) {}^{n-1}C_x p^x q^{n-(x+1)} \\
&= v^{n-1}\sum_{x=a}^{n-1} \left(S_0 (u-d) \times u^x d^{(n-1)-x} - k + k\right) {}^{n-1}C_x p^x q^{(n-1)-x} \\
&= S_0 (u-d) v^{n-1}\sum_{x=a}^{n-1} {}^{n-1}C_x (pu)^x (qd)^{(n-1)-x}
\end{aligned}
$$

The call option delta $h_{0,n}$ is the partial difference of f with respect to S, and the ratio of the two differences $h_{0,n} = \delta f_{1,n} / \delta S_1$

$$
h_{0,n} = \frac{f_u - f_d}{S_0(u-d)} = v^{n-1}\sum_{x=a}^{n-1} {}^{n-1}C_x (pu)^x (qd)^{(n-1)-x} = (vpu + vqd)^{n-1}\sum_{x=a}^{n-1} {}^{n-1}C_x \left(\frac{pu}{pu+qd}\right)^x \left(\frac{qd}{pu+qd}\right)^{(n-1)-x}
$$

i.e.,

$$
h_{0,n} = (vpu + vqd)^{n-1} \bar{F}_{B(n-1,\,p')}(a)
$$

where $p' = pu / (pu + qd)$ and $\bar{F}_{B(n-1,p')}(\cdot)$ is the complementary CMF of a binomial variable with parameters n and p'. For any time j the call option delta conditional on the spot price S_j is

$$
h_{j,n} = (vpu + vqd)^{n-j-1} \bar{F}_{B(n-j-1,p')}(a_{j,n}) \qquad \text{where } a_{j,n} = \log(k / S_j d^{n-j}) / \log(u/d)
$$

If the exercise price is not on the lattice then there is the additional term that would otherwise zero so

$$
f_u - f_d = v^{n-1}(S_0 u^a d^{n-a} - k) {}^{n-1}C_{a-1} p^{a-1} q^{n-a} + S_0 (u-d) v^{n-1}\sum_{x=a}^{n-1} {}^{n-1}C_x (pu)^x (qd)^{(n-1)-x}
$$

and the delta is

$$
\begin{aligned}
h_{0,n} &= v^{n-1}\sum_{x=a}^{n-1} {}^{n-1}C_x (pu)^x (qd)^{(n-1)-x} + v^{n-1}\frac{(S_0 u^a d^{n-a} - k)}{S_0(u-d)} {}^{n-1}C_{a-1} p^{a-1} q^{n-a} \\
&= (vpu + vqd)^{n-1}\sum_{x=a}^{n-1} {}^{n-1}C_x \left(\frac{pu}{pu+qd}\right)^x \left(\frac{qd}{pu+qd}\right)^{(n-1)-x} + v^{n-1}\frac{(S_0 u^a d^{n-a} - k)}{S_0(u-d)} {}^{n-1}C_{a-1} p^{a-1} q^{n-a} \\
&= (vpu + vqd)^{n-1} \bar{F}_{B(n-1,p')}(a) + v^{n-1}\frac{(S_0 u^a d^{n-a} - k)}{S_0(u-d)} {}^{n-1}C_{a-1} p^{a-1} q^{n-a}
\end{aligned}
$$

This additional term is small since $S_0u^ad^{n-a} \approx k$ if it is not exactly equal and is the product of three small terms, two of which between zero and one ($0 < v^{n-1} < 1$ and $0 < {}^{n-1}C_{a-1}p^{a-1}q^{n-a} < 1$). For a put option, if the spot price of the underlying market asset appreciates with an upward step to S_0u the put option premium becomes

$$\begin{aligned} f_u &= v^{n-1}\sum_{x=0}^{n-1} \max\left(k - S_0u \times u^x d^{n-1-x}, 0\right) {}^{n-1}C_x p^x q^{n-1-x} \\ &= v^{n-1}\sum_{x=0}^{a-1} \left(k - S_0u^{x+1}d^{n-1-x}\right) {}^{n-1}C_x p^x q^{n-1-x} \end{aligned}$$

If the spot price of the underlying market asset depreciates with a downward step to S_0d, the put option premium changes to

$$\begin{aligned} f_d &= v^{n-1}\sum_{x=0}^{n-1} \max\left(k - S_0d \times u^x d^{n-1-x}, 0\right)^{n-1} C_x p^x q^{n-1-x} \\ &= v^{n-1}\sum_{x=0}^{a} \left(k - S_0u^x d^{n-x}\right) {}^{n-1}C_x p^x q^{n-1-x} \\ &= v^{n-1}\left(k - S_0u^a d^{n-a}\right) {}^{n-1}C_a p^a q^{n-1-a} + v^{n-1}\sum_{x=0}^{a-1} (k - S_0u^x d^{n-x})^{n-1} C_x p^x q^{n-1-x} \\ &= v^{n-1}\sum_{x=0}^{a-1} \left(k - S_0u^x d^{n-x}\right) {}^{n-1}C_x p^x q^{n-1-x} \end{aligned}$$

where it is assumed that the exercise price is on the lattice to keep the maths concise. The difference in the put option premium for the two alternative forward steps and two possible values of the premium is

$$\begin{aligned} f_u - f_d &= v^{n-1}\sum_{x=0}^{a-1} \left(k - S_0u^{x+1}d^{n-1-x}\right) {}^{n-1}C_x p^x q^{n-1-x} - v^{n-1}\sum_{x=0}^{a-1} \left(k - S_0u^x d^{n-x}\right) {}^{n-1}C_x p^x q^{n-1-x} \\ &= v^{n-1}\sum_{j=0}^{a-1} \left(k - k - S_0(u-d)u^x d^{n-1-x}\right) {}^{n-1}C_x p^x q^{n-1-x} \\ &= -S_0(u-d)v^{n-1}\sum_{x=0}^{a-1} {}^{n-1}C_x (pu)^x (qd)^{(n-1)-x} \end{aligned}$$

so that the initial put option delta at time zero is

$$h_{0,n} = \frac{f_u - f_d}{S_0(u-d)} = -\sum_{x=0}^{a-1} {}^{n-1}C_x (vpu)^x (vqd)^{(n-1)-x} = -(vpu + vqd)^{n-1}\sum_{x=0}^{a-1} {}^{n-1}C_x \left(\frac{pu}{pu+qd}\right)^x \left(\frac{qd}{pu+qd}\right)^{n-1-x}$$

i.e.,

$$h_{0,n} = -(vpu + vqd)^{n-1} F_{B(n-1,p')}(a-1)$$

where $p' = pu / (pu + qd)$ and $F_{B(n-1,p')}(\cdot)$ is the CMF of a binomial variable with parameters n and p'. For any time j the put option delta conditional on the spot price S_j is

$$h_{j,n} = -(vpu + vqd)^{n-j-1} F_{B(n-j-1,p')}(a_{j,n} - 1) \qquad \text{where } a_{j,n} = \log(k / S_j d^{n-j}) / \log(u/d)$$

If the exercise price is not on the lattice

$$\begin{aligned} f_u - f_d &= v^{n-1}\sum_{x=0}^{a-1} \left(k - S_0u^{x+1}d^{n-1-x}\right) {}^{n-1}C_x p^x q^{n-1-x} \\ &\quad - v^{n-1}\sum_{x=0}^{a-1} \left(k - S_0u^x d^{n-x}\right) {}^{n-1}C_x p^x q^{n-1-x} - v^{n-1}\left(k - S_0u^a d^{n-a}\right) {}^{n-1}C_a p^a q^{n-1-a} \\ &= v^{n-1}\sum_{j=0}^{a-1} \left(k - k - S_0(u-d)u^x d^{n-1-x}\right) {}^{n-1}C_x p^x q^{n-1-x} - v^{n-1}\left(k - S_0u^a d^{n-a}\right) {}^{n-1}C_a p^a q^{n-1-a} \\ &= -S_0(u-d)v^{n-1}\sum_{x=0}^{a-1} {}^{n-1}C_x (pu)^x (qd)^{(n-1)-x} - v^{n-1}\left(k - S_0u^a d^{n-a}\right) {}^{n-1}C_a p^a q^{n-1-a} \end{aligned}$$

so that there is the additional trivial term that would otherwise zero

$$h_{0,n} = -(vpu + vqd)^{n-1} F_{B(n-1,p')}(a-1) - v^{n-1}\frac{\left(k - S_0u^a d^{n-a}\right)}{S_0(u-d)} {}^{n-1}C_a p^a q^{n-1-a}$$

The deltas can be calculated by reference to the formulae or the lattice of the deltas can be obtained from the ratio of the vertically differenced lattice of the premium f to that of the market asset S. This thesis has also derived the **recursions for the option delta (this page), as well as recursions for gamma (next page) and expectations of the delta position in the replicating asset and the replicating portfolio**. These are derived from first principles when the recursion for the premium is derived by reference to the transition PMF. This means

that the lattice of the deltas can be obtained by chaining back from the forward boundary at option expiration with the recursion $h_0 = vpuh_u + vqdh_d$ which is that for the option premium with the additional return factors in the coefficients. In full notation for an arbitrary step *j* and forward variables at step *n*, this is

$$h_{j,n}(k \mid S_j) = vpuh_{j+1,n}(k \mid S_j u) + vqdh_{j+1,n}(k \mid S_j d)$$

the lattice for which has one less step than that of the option since it is an end-of-step quantity and a quantity that depends on the premium differences whereby one step of any lattice is lost for every difference δ.

Backwards recurrence for the option delta h

Thes backwards recurrence for the option delta is now derived. From the backwards recurrence of the option premium

$$f_0 = vpf_u + vqf_d$$

if the spot price moves upwards, then the recurrence becomes

$$f_u = vpf_{uu} + vqf_{ud}$$

and similarly, if the spot price moves downwards the recurrence becomes

$$f_d = vpf_{ud} + vqf_{dd}$$

Subtracting the two

$$f_u - f_d = vp(f_{uu} - f_{ud}) + vq(f_{ud} - f_{dd})$$

and dividing by $S_0 u - S_0 d$ for

$$\frac{f_u - f_d}{S_0(u-d)} = vp\frac{(f_{uu} - f_{ud})}{S_0(u-d)} + vq\frac{(f_{ud} - f_{dd})}{S_0(u-d)}$$

i.e.,

$$\frac{f_u - f_d}{S_0(u-d)} = vup\frac{(f_{uu} - f_{ud})}{S_0(u^2 - ud)} + vdq\frac{(f_{ud} - f_{dd})}{S_0(ud - d^2)} \qquad \frac{f_u - f_d}{S_u - S_d} = vup\frac{f_{uu} - f_{ud}}{S_{uu} - S_{ud}} + vdq\frac{f_{ud} - f_{dd}}{S_{ud} - S_{dd}}$$

which can be written as

$$h_0 = vpuh_u + vqdh_d$$

which is the backwards recurrence of the option delta. This is the same as that for the option premium with the difference, aside from the boundary conditions, that the return ratios also appear in the coefficients. In full notation for an arbitrary step *j* and forward variables at step *n*, this is

$$h_{j,n}(k \mid S_j) = vpuh_{j+1,n}(k \mid S_j u) + vqdh_{j+1,n}(k \mid S_j d)$$

subject to the boundary from where the backwards recursion starts

$$h_{n-1,n}(k \mid S_{n-1}) = \frac{f_{n,n}(k \mid S_{n-1}u) - f_{n,n}(k \mid S_{n-1}d)}{S_{n-1}u - S_{n-1}d} \qquad \text{for} \qquad \begin{matrix} f_{n,n}(k \mid S_n) = \max(S_n - k, 0) \text{ for a call to buy} \\ f_{n,n}(k \mid S_n) = \max(k - S_n, 0) \text{ for a call to sell} \end{matrix}$$

as per the dependence on the recurrence for the premium *f* itself, and where there is one notional update which enters into the cashflows at option expiry for two paths whereby an exercise price is between an inthemoney and outofthemoney exercise when the exercise price is not on the lattice of the forward variables [when *k* is sandwiched between two of the *n* + 1 nodes on the terminus of the asset lattice rather than exactly on one of them]. This means that a closing out portfolio value, which is ordinarily *k* if the option expires inthemoney and zero if the option expires outofthemoney, additionally has two paths for which the closing out value of the portfolio is between zero and *k* due to the penultimate delta which then must either be shorted to zero or traded up to 1 as part of closing out the portfolio. Note that the lattice for the delta is for end-of-node values whereas that for the premium is for start-of-node values.

***Backwards recurrence for the option gamma* Γ**

Defining the option gamma as the partial rate of change in the option delta [for multinomial walks the definition is slightly different, and is the second order partial rate of change of the premium, which differs slightly in denominator]. With this definition

$$\Gamma_0 = \frac{h_u - h_d}{S_u - S_d}$$

From the backward recurrence of the delta, $h_0 = vpuh_u + vqdh_d$, if the backward walk variable moves up, then the backward recurrence for the option delta becomes $h_u = vpuh_{uu} + vqdh_{ud}$ whilst if the backward variable moves down then backward recurrence for the option delta becomes $h_d = vpuh_{ud} + vqdh_{dd}$. Subtracting the two for

$$h_u - h_d = vpu(h_{uu} - h_{ud}) + vqd(h_{ud} - h_{dd})$$

and dividing through by $S_0(u-d)$ for

$$\frac{h_u - h_d}{S_0(u-d)} = vpu\frac{(h_{uu} - h_{ud})}{S_0(u-d)} + vqd\frac{(h_{ud} - h_{dd})}{S_0(u-d)}$$

i.e.,

$$\frac{h_u - h_d}{S_0(u-d)} = vpu^2\frac{(h_{uu} - h_{ud})}{S_0(u^2-ud)} + vqd^2\frac{(h_{ud} - h_{dd})}{S_0(ud-d^2)} \qquad \frac{h_u - h_d}{S_u - S_d} = vpu^2\frac{h_{uu} - h_{ud}}{S_{uu} - S_{ud}} + vqd^2\frac{h_{ud} - h_{dd}}{S_{ud} - S_{dd}}$$

which can then be written in terms of the option gamma as defined as the rate of change in the option delta with respect to the market asset

$$\Gamma_0 = vpu^2\Gamma_u + vqd^2\Gamma_d$$

In full notation for a backward variable at step *j*, this is

$$\Gamma_{j,n}(k \mid S_j) = vpu^2\Gamma_{j+1,n}(k \mid S_j u) + vqd^2\Gamma_{j+1,n}(k \mid S_j d)$$

Writing the recursion for option delta in terms of the option gamma, for an upstep

$$\begin{aligned} h_0 &= vpuh_u + vqd(h_d - h_u + h_u) \\ &= vpuh_u + vqdh_u - vqd(h_u - h_d) \\ &= v(pu + qd)h_u - vqd(u-d)S_0\Gamma_0 \end{aligned}$$

and for a downstep

$$\begin{aligned} h_0 &= vpu(h_u - h_d + h_d) + vqdh_d \\ &= vpu(h_u - h_d) + vpuh_d + vqdh_d \\ &= v(pu + qd)h_d + vpu(u-d)S_0\Gamma_0 \end{aligned}$$

so that

(1) $h_u = \dfrac{(1+r)h_0 + qd(u-d)S_0\Gamma_0}{(pu+qd)}$

(2) $h_d = \dfrac{(1+r)h_0 - pu(u-d)S_0\Gamma_0}{(pu+qd)}$

This proves that each delta rebalance therefore contains gamma [independently of the precise definition of gamma] so that any missed delta rebalance means that gamma enters into the portfolio cash flows. This term cannot be neutralized and accumulates with every missed delta rebalance invalidating the replicating arguments, i.e., void from any one single missed binomial fluctuation; the evidence for this for a debased portfolio is a non-zero NPV of replicating cashflows from stochastic variation [**gamma variation**[2]] and nothing other than a

[2] as in gamma radiation, from nuclear physics, on the high frequency end of the electro-magnetic spectrum; given that mathematics is Universal, this is consistent with my contention that the stochastic variation of the Universe emanates [with nested basis functions on *S*] from the structure of matter.

skewed and convoluted option payoff profile. Options proper are always risky, with no exceptions. **Options without risk are not options**; more particularly, **options without gamma are pseudo options**. Options always require probability theory.

Table 1: *f*, *h*, Γ

Replicating Portfolio Functions		Backwards Deconvolution Recursion
p_S,	**transition PMF**	$p_{S_n\|S_0}(x,n;S_0,j)=\sum_{y=d,u}\ p_X(y)\,p_{S_n\|S_0}(x,n;S_0\times y,j+1)$
f,	**option**	$f(k,n;S_0,j)=v\sum_{y=d,u}\ p_X(y)\,f(k,n;S_0\times y,j+1)$
h,	**option delta**	$h(k,n;S_0,j)=v\sum_{y=d,u}\ y\times p_X(y)h(k,n;S_0\times y,j+1)$
Γ,	**option gamma**	$\Gamma(k,n;S_0,j)=v\sum_{y=d,u}\ y^2\times p_X(y)\Gamma(k,n;S_0\times y,j+1)$

where $X\sim B(1,p)$ so that $p_X(x)=p^xq^{1-x}\qquad x=0,1$

Table 2: *f*, *h*, Γ (shorthand notation)

Replicating Portfolio Functions		Backwards Deconvolution Recursion
f,	**option**	$f_0=vpf_u+vqf_d$
h,	**option delta**	$h_0=vpuh_u+vqdh_d$
Γ,	**option gamma**	$\Gamma_0=vpu^2\Gamma_u+vqd^2\Gamma_d$

Solving the recurrence for the delta of the option premium accurate for the market asset to deliver

Similarly to the option premium, the backwards recursion for the option delta can be solved by unfolding it over the backward variables to the forward variables at option expiration. From $h_0=vpuh_u+vqdh_d$, in one steps time, this either

$h_u=vpuh_{uu}+vqdh_{ud}$ if the spot increases to S_0u and
$h_d=vpuh_{du}+vqdh_{dd}$ if the spot decreases to S_0d

Similarly, in two steps time

$h_{uu}=vpuh_{uuu}+vqdh_{uud}$ if the spot moves to S_0uu with the second step
$h_{ud}=vpuh_{udu}+vqdh_{udd}$ if the spot moves to S_0ud with the second step
$h_{dd}=vpuh_{ddu}+vqdh_{ddd}$ if the spot moves to S_0dd with the second step

and so forth. Unfolding the recurrence to the forward variables

$$\begin{aligned}
h_0&=vpuh_u+vqdh_d\\
&=vpu\left(vpuh_{uu}+vqdh_{ud}\right)+vqd\left(vpuh_{ud}+vqdh_{dd}\right)\\
&=v^2p^2u^2h_{uu}+2v^2pqudh_{ud}+v^2q^2d^2h_{dd}\\
&=v^2p^2u^2\left(puh_{uuu}+qdh_{uud}\right)+2v^2puqd\left(puh_{udu}+qdh_{udd}\right)+v^2q^2d^2\left(puh_{ddu}+qdh_{ddd}\right)\\
&=v^3p^3u^3h_{uuu}+3v^3p^2u^2qdh_{uud}+3v^3puq^2d^2h_{udd}+v^3q^3d^3h_{ddd}\\
&\ldots\\
&=v^{n-1}\sum_{x=0}^{n-1}\ {}_{n-1}C_x(pu)^x(qd)^{n-1-x}h_{u^xd^{n-1-x}}
\end{aligned}$$

At the boundary at the forward variables, the delta is one for that part of the range where the option is inthemoney and zero for that part where the option is outofthemoney. If the exercise price is not on the lattice, then the delta by which the forward step is inthemoney and the downward step is outofthemoney is between zero and one. This occurs at $x=a-1$ for a call and $x=a$ for a put where a was defined in this book when the premium was introduced as the nominal log-standardized value of the exercise price for **(1)** the minimum num-

ber of upsteps for a call option to expire inthemoney and **(2)** the maximum number of upsteps for a put option to expire inthemoney. For a call option, if the exercise price is on the sample space of the forward variables, then

$$h_{u^x d^{n-1-x}} = \begin{cases} 1 & for\ x > a-1 \\ 0 & for\ x < a-1 \end{cases}$$

so that

$$\begin{aligned} h_0 &= v^{n-1} \sum_{x=a}^{n-1} {}^{n-1}C_x (pu)^x (qd)^{n-1-x} \times 1 \\ &= (vpu + vqd)^{n-1} \bar{F}_{B(n-1,p')}(a) \end{aligned}$$

as required, where $\bar{F}_{B(n-1,p')}(\cdot)$ is a complementary CMF of a binomial variable with index $n-1$ and probability $pu/(pu+qd)$ and where $a = \log(k/S_0 d^n)/\log(u/d)$. If the exercise price is not on the sample space of the forward variables, then there is the numerical term for the penultimate delta for the last step which is then inthemoney for an upstep and outofthemoney for a downstep. The node just before expiry is $S_0 u^{a-1} d^{(n-1)-(a-1)} = S_0 u^{a-1} d^{n-a}$ which either moves up into the money at $S_0 u \times u^{a-1} d^{n-a} = S_0 u^a d^{n-a}$ or down outofthemoney at $S_0 d \times u^{a-1} d^{n-a} = S_0 u^{a-1} d^{n-a+1}$ so

$$h_{u^{a-1} d^{n-a}} = \frac{f_{n,n}(k \mid S_0 u^a d^{n-a}) - f_{n-1,n}(k \mid S_0 u^{a-1} d^{n-a+1})}{S_0 u^a d^{n-a} - S_0 u^{a-1} d^{n-a+1}} = \frac{f_{n,n}(k \mid S_0 u^a d^{n-a})}{S_0 u^{a-1} d^{n-a}(u-d)} = \frac{(S_0 u^a d^{n-a} - k)}{S_0 u^{a-1} d^{n-a}(u-d)}$$

and therefore

$$h_{u^x d^{n-1-x}} = \begin{cases} 1 & for\ x > a-1 \\ \dfrac{(S_0 u^a d^{n-a} - k)}{S_0 u^{a-1} d^{n-a}(u-d)} & for\ x = a-1 \\ 0 & for\ x < a-1 \end{cases}$$

Hence

$$\begin{aligned} h_0 &= v^{n-1} \sum_{x=a}^{n-1} {}^{n-1}C_x (pu)^x (qd)^{n-1-x} \times 1 + v^{n-1} \sum_{x=a-1}^{a-1} {}^{n-1}C_x (pu)^x (qd)^{n-1-x} \frac{(S_0 u^a d^{n-a} - k)}{S_0 u^{a-1} d^{n-a}(u-d)} \\ &= (vpu + vqd)^{n-1} \bar{F}_{B(n-1,p')}(a) + v^{n-1} \frac{(S_0 u^a d^{n-a} - k)}{S_0 (u-d)} {}^{n-1}C_{a-1} p^{a-1} q^{n-a} \end{aligned}$$

with the inclusion of the trivial term [in light of the expense in algebra]. For a put option with the exercise price on the sample space of the forward variables

$$h_{u^x d^{n-1-x}} = \begin{cases} 0 & for\ x > a \\ -1 & for\ x < a \end{cases}$$

so that

$$\begin{aligned} h_0 &= -v^{n-1} \sum_{x=0}^{a-1} {}^{n-1}C_x (pu)^x (qd)^{n-1-x} \times 1 \\ &= -(vpu + vqd)^{n-1} F_{B(n-1,p')}(a-1) \end{aligned}$$

as required, where $F_{B(n-1,p')}(\cdot)$ is a CMF of a binomial variable with index $n-1$ and probability $pu/(pu+qd)$ and where $a = \log(k/S_0 d^n)/\log(u/d)$. If the exercise price is not on the sample space of the forward variables, then there is the numerical term for the penultimate delta for which an upstep is outofthemoney and for which a downstep is inthemoney. The node just before expiry is $S_0 u^a d^{(n-1)-a} = S_0 u^a d^{n-a-1}$ which moves up outofthemoney at $S_0 u \times u^a d^{n-a-1} = S_0 u^{a+1} d^{n-a-1}$ or down into the money at $S_0 d \times u^a d^{n-a-1} = S_0 u^a d^{n-a}$ since a put is a deferred optional short trade. So

$$h_{u^a d^{n-a-1}} = \frac{f_{n,n}(k \mid S_0 u^{a+1} d^{n-a-1}) - f_{n-1,n}(k \mid S_0 u^a d^{n-a})}{S_0 u^{a+1} d^{n-a-1} - S_0 u^a d^{n-a}} = -\frac{f_{n-1,n}(k \mid S_0 u^a d^{n-a})}{S_0 u^a d^{n-a-1}(u-d)} = -\frac{(k - S_0 u^a d^{n-a})}{S_0 u^{a-1} d^{n-a}(u-d)}$$

therefore

$$h_{u^x d^{n-1-x}} = \begin{cases} 0 & for\ x > a \\ -\dfrac{(k - S_0 u^a d^{n-a})}{S_0 u^{a-1} d^{n-a}(u-d)} & for\ x = a \\ -1 & for\ x < a \end{cases}$$

Hence

$$h_0 = -v^{n-1}\sum_{x=0}^{a-1} {}^{n-1}C_x (pu)^x (qd)^{n-1-x} \times 1 - v^{n-1}\sum_{x=a}^{a} {}^{n-1}C_x (pu)^x (qd)^{n-1-x} \frac{(k - S_0 u^a d^{n-a})}{S_0 u^a d^{n-a-1}(u-d)}$$

$$= -(vpu + vqd)^{n-1} F_{B(n-1,p')}(a-1) - v^{n-1}\frac{(k - S_0 u^a d^{n-a})}{S_0 (u-d)} {}^{n-1}C_a p^a q^{n-a-1}$$

as previously.

Components of Change in the Option Replicating Portfolio for the Binomial Market Asset S

This section discusses the components of change in the replicating portfolio by reference to changes between the nodes and at the nodes of the binomial lattice, and how the partition of the changes of the replicating portfolio between and at the nodes are either value changes (in S and f) or quantity changes (in h).

Taking an offsetting position in a quantity of the underlying asset to replicate and nullify a short position in an option requires taking a position in the market asset, and continually maintaining that position, by buying and selling the market asset by reference to changes in the [momentary[3]] hedge ratios to replicate the option premium [and ultimately the option]. For a binomial walk, delta is sufficient to offset the option. Then, for a binomial delta-neutralized replicating portfolio, the initial delta h_0 is the quantity of the market asset purchased to establish the portfolio for a long position in the binomial asset to offset the short option. The net value of the position in the offsetting market asset and the option is the value of the replicating portfolio, and the net cost to establish the portfolio, i.e., the cost of taking the position in the market asset $h_{0,n} S_0$, less the revenue from writing the option $f_{0,n}$, both at the prevailing prices $h_{0,n} S_0 - f_{0,n}$ (the spot price and the premium calculated at the spot price). For a call option proper, the purchase of the asset always exceeds the revenue from the premium. The value of delta and the hedge portfolio at time *j*, conditional on the time *j* spot price S_j, are $h_{j,n}$ and $\Pi_{j,n}$. The value of the replicating portfolio at time *j* is

$$\Pi_{j,n} = h_{j,n} S_j - f_{j,n}$$

Asset/Liability		**Quantity**	**Value**
market asset,	S_j	$+\ h_{j,n}$	$+\ h_{j,n} S_j$
option,	$f_{j,n}$	-1	$-f_{j,n}$
total,	$\Pi_{j,n}$		$h_{j,n} S_j - f_{j,n}$

By the j^{th} step, the value of the portfolio is a result of **(1)** the value changes of the quantities in the portfolio, which comprise market movements in the asset *S* and in the option premium *f*, plus **(2)** the quantity changes (in *h*), for the additional units of the market asset bought or sold by reference to changes in delta *h* to rebalance the replicating portfolio to keep the offsetting replicating delta matched to that of the option delta and maintain an unbroken replication [for a killed option and cashflows without gamma and higher order variation]. The value of the hedge portfolio at time *j* can then be rearranged into the form

$$\Pi_{j,n} = (h_{0,n} + h_{j,n} - h_{0,n}) S_j - f_{j,n}$$
$$= (h_{0,n} + \sum_{i=0}^{j-1} \Delta h_{i,n}) S_{j,n} - f_{j,n}$$

where the delta difference Δ of the hedge ratio *h* acts on the backward time variable *j* for a fixed forward time variable and option expiry *n*. This shows that the total quantity changes the portfolio undergoes by the j^{th} step is

[3] instantaneous in continuous time, and must be sourced continuously otherwise the replication is void.

the total net quantity of the market asset purchased or sold to rebalance the portfolio, which is the sum of the delta changes

$$\sum_{i=0}^{j-1} \Delta h_{i,n} = \Delta h_{0,n} + \Delta h_{1,n} + ... + \Delta h_{j-1,n}$$

over and above the quantity purchased to establish the portfolio, $h_{0,n}$. These are the changes that occur at the binomial lattice nodes, at the end of every step of the walk for *S*, and for which are due at the beginning of every step as an outstanding contingent payment to be made at the end of the step. Each of these contingent payments must be borne from equity, otherwise the portfolio does not remain delta-neutral and the portfolio cashflows then depend on gamma and higher order variation, i.e., if delta is not rebalanced following any change in the value of the market asset *S* to keep it matched with the option delta. The replicating delta $h_{j,n}$ are set at the beginning of any step by reference to the end of the step for which the next rebalance is made so the total cost of the purchases by the j^{th} step is

$$\sum_{i=0}^{j-1} S_{i+1}\Delta h_{i,n} = S_1\Delta h_{0,n} + S_2\Delta h_{1,n} + ... + S_j\Delta h_{j-1,n}$$

where the S_{i+1} as well as $h_{i+1,n}$ in $\Delta h_{i+1,n} = h_{i+1,n} - h_{i,n}$ are unknown when they are outstanding at the beginning of step *i* as conditioned on S_i and pose contingent risk dependent upon the binomial basis ΔS_i. The value of these contingent buys and sells discounted to the backward variable at the time the option is written is then

$$\sum_{i=0}^{j-1} v^{i+1}S_{i+1}\Delta h_{i,n} = vS_1\Delta h_{0,n} + v^2S_2\Delta h_{1,n} + ... + v^jS_j\Delta h_{j-1,n}$$

where *v* is at the single period effective cash rate *r*. At expiry of the option at the forward variables and immediately following closing out the hedge portfolio and delivering the asset *S* if the option is inthemoney, at the end of step *n* of the binomial walk, this sum for $j = n$ contains the total present value of the hedging rebalancing costs

$$\sum_{j=0}^{n-1} v^{j+1}S_{j+1}\Delta h_{j,n} = vS_1\Delta h_{0,n} + v^2S_2\Delta h_{1,n} + ... + v^nS_n\Delta h_{n-1,n}$$

which is the present value of each of the delta rebalances and total hedging cost, net of the initial purchase $S_0h_{0,n}$, and is a random walk born out of equity. More precisely, it is a derivative of an exponential diffusion so is something that mimics an inverse diffusion [which is noisier than white noise]. A delta-neutralized portfolio only remains delta-neutral for one single moment [step] unless it is rebalanced at the end of that step. This is the condition that **every market fluctuation must be neutralized** if the replicating portfolio is to maintain an unbroken replication, including when limits are taken whereby the number of nodes and rebalances becomes infinite for any finite term to maturity. This necessarily means high frequency. If any one of these is missed, the offsetting delta does not match that of the option delta and ΔS enters into the cashflows so that the portfolio ceases to be delta-neutral and the replication is broken and **the delta-neutral arguments are void. This amounts to accumulated gamma and higher terms of variation introduced into the replicating portfolio and cashflows, since missed delta terms accumulate into gamma terms and these gamma terms then accumulate into higher order terms of stochastic variation. These terms [defined by this thesis as gamma variation] cannot be neutralized.** Anything short of consistent high frequency trading means that the replication is void.

The forward difference in the asset component $S_j\, h_{j,n}$ of the replicating portfolio can be expressed in three equivalent forms

$$\begin{aligned}\Delta(h_{j,n}\, S_j) &= h_{j+1,n}\, S_{j+1} - h_{j,n}\, S_j \\ &= S_j\Delta h_{j,n} + h_{j+1,n}\,\Delta S_j \\ &= h_{j,n}\,\Delta S_j + S_{j+1}\Delta h_{j,n}\end{aligned}$$

The change in the offsetting asset $h_{j,n}\, S_j$ of the replicating portfolio has therefore been partitioned into two components, rather than letting it change as a single function. It is therefore a fact that delta *h* and the market asset *S* both change at the same time and therefore changes in *S* must be viewed in conjunction with changes in *h*. For the particular form of the discrete product rule $\Delta(h_{j,n}\, S_j) = h_{j,n}\,\Delta S_j + S_{j+1}\Delta h_{j,n}$, the first of the two components of the split of the total difference $h_{j,n}\,\Delta S_j$ is that used to offset the option which leaves the second component $S_{j+1}\Delta h_{j,n}$ which comprises the delta update to keep the offsetting deltas in proportion and maintain the replication. The size of this delta update due is unknown at the start of the vertices and the position in the delta-neutralized portfolio is only free from stochastic variation if **(1)** this delta update is paid for in advance since it is

outstanding at the beginning of any step or **(2)** the portfolio is fully immunized from market dependent cash flow requiring the asset value to exactly match the option premium or the option to be a pseudo-option (i.e., $a = 0$). Full immunization which occurs for options proper only if $k = 0$ or $a = 0$ (**for binomial variables, whenever there is zero probability that the option will expire outofthemoney**) so that the value of the market dependent cashflows of the asset exactly matches those of the option [in addition to just the offsetting deltas]. In the case of a call, the portfolio is only immunized at the backward variables, for example, if the premium revenue is sufficient to fund the purchase of the offsetting asset and the deltas are deterministic functions. If the portfolio is not fully immunized, the size of the delta update due depends on binomial CMFs (and consequently depends on ΔS_j) and is hence unknown and cannot be paid for in advance since it is a random variable. The random variable of the update $S_1\Delta h_0 = S_1(h_1 - h_0)$ has mean

$$vE\left(S_1(h_1 - h_0) \mid S_0\right) = -(puv + qdv - 1)S_0 h_0$$

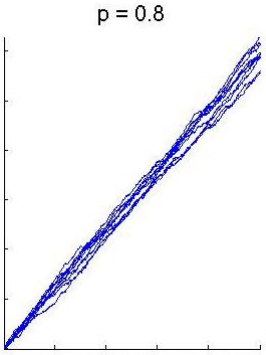
and variance

$$Var\left(S_1(h_1 - h_0) \mid S_0\right) = S_0{}^2 pq\left(u(h_u - h_0) - d(h_d - h_0)\right)^2$$

for which the proof is omitted because is it long and unpleasant. This expression is similar to that [the variance] of the market asset $Var\left(S_1 \mid S_0\right) = S_0{}^2 pq\left(u - d\right)^2$ and option $Var\left(f_1 \mid S_0\right) = h_0^2 S_0{}^2 pq\left(u - d\right)^2$. Because of this (the variation in $S_{j+1}\Delta h_{j,n}$), maintaining an option replicating portfolio is never risk free and there is always ΔS_j dependent risk outstanding if a replication is to remain unbroken and free from gamma and higher order variation. Risk-neutral arguments conveniently bypass this fact by pretending that delta is constant for which it is then possible to say that $\Delta(h\, S_j) = h\, \Delta S_j$ which is **not rigorous nor is it correct and only leads to incorrect deductions**. Both parts to the change $\Delta(h\, S_j)$ must be viewed in conjunction with one another. With the offsetting asset change as $\Delta(h_{j,n} S_j) = h_{j,n}\Delta S_j + S_{j+1}\Delta h_{j,n}$, the forward increment for the total change in the portfolio from time j to time $j + 1$ is composed of an appreciation in the delta quantity $h_{j,n}$ units of the market asset, plus the purchase of $\Delta h_{j,n}$ additional units of the asset at price S_{j+1} , less the change in the value of the option premium;

$$\begin{aligned}
\Delta\Pi_{j,n} &= \Delta(h_{j,n} S_j - f_{j,n}) \\
&= \Delta(h_{j,n} S_j) - \Delta f_{j,n} \\
&= h_{j,n}\Delta S_{j,n} + S_{j+1}\Delta h_{j,n} - \Delta f_{j,n} \\
&= (h_{j,n}\Delta S_j - \Delta f_{j,n}) + S_{j+1}\Delta h_{j,n}
\end{aligned}$$

and only by saying $h_{j,n}$ is constant does the risky term in S_{j+1} not feature in the expression for $\Delta\Pi_{j,n}$. **Delta changes and asset changes are two parts to one thing and one of them cannot be considered out of the context of the other for any option replicating portfolio.** It is wrong to say that $\Delta(h_{j,n} S_j) = h_{j,n}\Delta S_j + S_{j+1}\Delta h_{j,n} = h_{j,n}\Delta S_j$ unless and only unless h is constant or there is explicit recognition of the update that must be made at **each and every step** j from the time the option is written up to the time it expires; **not ignoring *h* means a rebalance due from equity at each and every equally spaced node and every step of the walk, which amounts to every single fluctuation in the market asset *S*; *n* steps to option expiration means *n* rebalances and trades in *S*.** This amounts to a **100% neutralization ratio** and is also essential to recognize for limiting arguments where the step size is made infinitesimal and the number of steps in the transition becomes infinite, i.e., subsidizing the debased price of Black and Scholes (1973) necessarily involves continuously maintained ultra-high frequency trading otherwise the arguments are completely void and the debased price is completely invalid; **if there are 10,000 steps in the limiting argument, for example [as a proxy to maintaining a continuous signal], then there must be 10,000 equally spaced rebalances, otherwise the replication is broken and replicating arguments for a delta-neutralized portfolio are void. Any missed rebalance means that the cashflows will have an accumulation of unhedged gamma and higher order terms of variation within the net cashflow, and result in nothing other than a skewed and convoluted option payoff profile of an indeterminate form**. If delta is recognized as dependent upon the basis risk and consequently risky, it must be recognized that a delta-neutral portfolio is not risk free whenever the portfolio does not achieve full immunization (for $k > 0$ or $\Pr(S_n > k \mid S_0) = 1$) since a random walk must be sourced from equity and, additionally, that **neutralizing delta is not sufficient to neutralizing market risk for any variable that is not binomial.**



The fact that delta is stochastic becomes 'all-encompassing' for any asset that is not binomial i.e., is not either underdispersed or two-valued risk (i.e., whenever the asset return is not like the simulated variable to the left; how many of the paths in the sample space of a walk with $p = 0.8$ represent a falling market?). In continuous time where the asset always differs from a binomial walk, skipping any finite number of an infinite number of mandatory rebalances (any finite number of an infinite number of market fluctuations) opens up the very real possibility of say, writing an option (say \$30) revenue and then purchasing an offsetting asset (say \$70 cost) for a net outlay of \$40. The next day, if the asset falls to zero (or something negligible below the exercise price, which happens and is a very real possibility) and does not recover, then every delta rebalance thereafter will result in a loss making trade. No matter how much trading the portfolio does, by reference to delta, it will not be possible to recover. There will then be a net loss of the cash used to establish the portfolio of \$70 less the \$30 premium revenue, i.e., a loss of \$40. This is not risk free because the portfolio was **not fully immunized from market dependent cashflow** which requires that the assets exactly match the liabilities which means that the portfolio value does not cost a dime to establish and consequently then, and only then, there is no value at risk (say \$30 outlay for \$30 revenue, and then both [asset and option] go to \$0, then there is no loss). A delta-neutral option portfolio in continuous time, for either the accurate premium or the debaseded premium, is ALWAYS subject to risk [and gamma] and only in the contrived binomial context is it possible to trade your way out of a falling market (because there are none [falling markets] for exponential diffusion for most values of *p*). In all other circumstances [including contrived trinomial risk], risk from delta updates as well as risk from missed rebalances and voided replicating arguments can and does eat into portfolio returns, leaving a portfolio that has the same market risk as any market variable and because the payoff profile becomes so skewed and convoluted from accumulated gamma (from a **0% neutralization ratio**, e.g., $\lim_{n\to\infty}(5/n)=0$ for 5 rebalances), it is never possible to tell exactly what that risk is apart from the fact that it is gamma variation, which is proved by reference to the backwards recursion for delta and gamma (on page 13).

■

After determining the return components of the portfolio net of the asset trading for the delta updates, by determining an expression for $h_{j,n}\Delta S_j - \Delta f_{j,n}$, the discrete portfolio changes are summed into the expression for the net present value (NPV) of the cashflows of the portfolio. The mathematical derivations for the expression for the portfolio change net of the delta update

$$h_{j,n}\frac{\Delta S_j}{\Delta j} - \frac{\Delta f_{j,n}}{\Delta j} = (pu + qd - 1)S_j h_{j,n} - r\, f_{j,n}$$

amounts to several alternative proofs that the portfolio returns are not fixed and flat at the cash rate unless the portfolio and premium are debased and that, instead, vary from period to period due to the variable composition of the portfolio rebalanced with maintaining delta for an interaction between the market rate and the cash rate in proportion to the ratio of *Sh* to *f* with **(1)** the market return $pu + qd - 1$ earned on the market asset *S* and **(2)** the cash rate *r* incurred on the short option *f*. The expression for $h_{j,n}\Delta S_{j,n} - \Delta f_{j,n} = (pu + qd - 1)S_j h_{j,n} - r\, f_{j,n}$ is deduced firstly by combinatorial methods and then by reference to discrete calculus and difference equations.

It is the viewpoint of my thesis that options are not redundant securities and that returns earned on the replicating asset is the return for **(1)** giving up the option (by flatlining it with the offsetting replicating asset) and **(2)** taking a pecuniary interest in the economic asset and a position on the right side of the market in trading *S*, which amounts to trading long for an excess returning asset or trading short for an under-returning asset; this is revenue earned for the financial service of making a market in the option and taking up the pecuniary interest in the economic asset for the deferred transaction of the derivative; long in *S* for a call and short in *S* for a put.

Introduction

From combinatorial arguments and discrete calculus with the backwards recurrence of the statistically unbiased option for the traded market asset *S*, the between-node-returns of the hedge portfolio are obtained and used to show that this results in a portfolio return that is not fixed and flat at the cash rate but instead an interaction between market return and the cash rate *r*. In particular, the market asset earns the market return $pu + qd - 1$ whilst the short option incurs the cash rate *r* from time decay in the discount function on the premium. The portfolio return is not fixed flat from period to period because of the changing composition of the portfolio with each rebalance made at the end of each step and market fluctuation to keep the offsetting delta matched to that

of the option. From the result $h_{j,n}\Delta S_{j,n} - \Delta f_{j,n} = (pu + qd - 1)S_j h_{j,n} - r\, f_{j,n}$, the components of portfolio change are summed into an expression for the discounted costs of the hedge portfolio.

It is hence proved that the statement "that delta neutral portfolios must earn the cash rate" is wrong (proof presented at the 2010 PhD Colloquium) and that portfolios containing and replicating the correct and accurate premium earn the market rate on the market asset *S* and incur the cash rate on the option. From this result and the supporting mathematics, it is proposed by this thesis that derivative markets fully recognize the market return of the spot markets from which the value of the derivative markets derive and depend and that, further, premiums be statistically accurate measures for the risk and return that trade within the derivative markets as extensions of that of the spot markets. It is asserted that *return* and *premium* in the context of derivatives are *the income from* and *the price of* **deferring a spot transaction in an economic asset**, i.e., a deferring a pecuniary interest in, and claim to the economic activity of, the underlying market asset *S*.

For the replicating portfolio of the binomial walk for the statistically accurate premium, this thesis proves by two approaches that the return components are **(1)** the market return on the asset and **(2)** the cash rate on the option

(1) by combinatorial arguments
(2) by the discrete calculus and difference equations

which amounts to deducing the equation for the total portfolio change $\Delta\Pi_{j,n}$ with respect to *j* for a single step in the walk for *S*

$$\frac{\Delta\Pi_{j,n}}{\Delta j} = (pu + qd - 1)S_j h_{j,n} - rf_{j,n} + S_{j+1}\frac{\Delta h_{j,n}}{\Delta j}$$

All of these proofs are original. For the second part of this book considering a continuously varying market asset *S*, combinatorial methods are not available to deduce the return components so they are deduced by calculus and differential equations (the PDE of the statistically unbiased premium, original to this thesis) and, where possible, as limits of the binomial results.

The Return Components of a Binomial Replicating Portfolio of the Statistically Unbiased Option for the Market Asset S: Combinatorial Proof

The return components by combinatorial arguments and proof that a delta-neutralized portfolio need not earn the cash rate, net of a delta update, is composed of three parts; the recursion for the premium, the expression for the delta-neutralized portfolio, and then the return components.

A. The option and its delta: Preliminaries and recursions

The statistically accurate premium for an option to call the market asset *S* to buy for *k* in *n* periods time is

$$\begin{aligned} f_0 &= v^n \sum_{x=0}^{n} \max(S_0 u^x d^{n-x} - k, 0)\, {}^nC_x p^x q^{n-x} \\ &= v^n \sum_{x=a}^{n} (S_0 u^x d^{n-x} - k)\, {}^nC_x p^x q^{n-x} \\ &= S_0 v^n \sum_{x=a}^{n} {}^nC_x (pu)^x (qd)^{n-x} - k v^n \sum_{x=a}^{n} {}^nC_x p^x q^{n-x} \end{aligned}$$

If the market asset *S* moves up, the premium becomes

$$\begin{aligned} f_u &= v^{n-1} \sum_{x=0}^{n-1} \max(S_0 u^{x+1} d^{n-1-x} - k, 0)\, {}^{n-1}C_x p^x q^{n-1-x} \\ &= v^{n-1} \sum_{x=a-1}^{n-1} (S_0 u^{x+1} d^{n-(x+1)} - k)\, {}^{n-1}C_x p^x q^{n-(x+1)} \\ &= \frac{1}{p} v^{n-1} \sum_{x=a-1}^{n-1} (S_0 u^{x+1} d^{n-(x+1)} - k)\, {}^{n-1}C_x p^{x+1} q^{n-(x+1)} \\ &= \frac{1}{p} v^{n-1} \sum_{x=a}^{n} (S_0 u^x d^{n-x} - k)\, {}^{n-1}C_{x-1} p^x q^{n-x} \end{aligned}$$

while if the market asset *S* moves down, the premium becomes

$$
\begin{aligned}
f_d &= v^{n-1}\sum_{x=0}^{n-1}\ \max(S_0u^xd^{n-x}-k,0)\ {}^{n-1}C_x p^x q^{n-1-x}\\
&= v^{n-1}\sum_{x=a}^{n-1}\ (S_0u^xd^{n-x}-k)\ {}^{n-1}C_x p^x q^{n-(x+1)}\\
&= \frac{1}{q}v^{n-1}\sum_{x=a}^{n-1}\ (S_0u^xd^{n-x}-k)\ {}^{n-1}C_x p^x q^{n-x}\\
&= \frac{1}{q}v^{n-1}\sum_{x=a}^{n}\ (S_0u^xd^{n-x}-k)\ {}^{n-1}C_x p^x q^{n-x}
\end{aligned}
$$

since ${}^{n}C_x = 0$ for $x<0$ as well as $x>n$ and is, incidentally, the reason why the binomial series terminates and successive rows form the triangular shape of Pascal's triangle. By combinatorial arguments, the fundamental relation of the lattice of the option premium is the recursive expectation

$$
\begin{aligned}
vpf_u + vqf_d &= vp\left(\frac{1}{p}v^{n-1}\sum_{x=a}^{n}\ (S_0u^xd^{n-x}-k)\ {}^{n-1}C_{x-1}p^xq^{n-x}\right)+vq\left(\frac{1}{q}v^{n-1}\sum_{x=a}^{n}\ (S_0u^xd^{n-x}-k)\ {}^{n-1}C_x p^xq^{n-x}\right)\\
&= v^n\sum_{x=a}^{n}\ (S_0u^xd^{n-x}-k)\ {}^{n-1}C_{x-1}p^xq^{n-x} + v^n\sum_{x=a}^{n}\ (S_0u^xd^{n-x}-k)\ {}^{n-1}C_x p^xq^{n-x}\\
&= v^n\sum_{x=a}^{n}\ (S_0u^xd^{n-x}-k)({}^{n-1}C_{x-1}+{}^{n-1}C_x)p^xq^{n-x}\\
&= v^n\sum_{x=a}^{n}\ (S_0u^xd^{n-x}-k)\ {}^{n}C_x p^xq^{n-x}\\
&= f_0
\end{aligned}
$$

since ${}^{n-1}C_{x-1}+{}^{n-1}C_x = {}^{n}C_x$, i.e.,

$$f_0 = vpf_u + vqf_d$$

which is proved again by first principles with reference to the PMF of the walk generator and walk transition distribution (starting on page 29). This recursion can show what the total change in the option premium is for any binomial step

$$
\begin{aligned}
(1+r)f_0 &= pf_u + q(f_d - f_u + f_u) & (1+r)f_0 &= p(f_u - f_d + f_d) + qf_d\\
(1+r)f_0 &= f_u - q(f_u - f_d) & (1+r)f_0 &= ph_0S_0(u-d) + f_d\\
qS_0(u-d)h_0 &= f_u - (1+r)f_0 & (1+r)f_0 - f_d &= ph_0S_0(u-d)
\end{aligned}
$$

i.e.,

$f_u = (1+r)f_0 + qS_0(u-d)h_0$ if the underlying spot price S_0 appreciates to S_0u and
$f_d = (1+r)f_0 - pS_0(u-d)h_0$ if the underlying spot price S_0 depreciates to S_0d

so that there is an in increase the option premium at the cash rate as a result of **time decay in the discount factor** plus a binomial increase or decrease in proportion to the option delta and the spot value of the asset; an increase of $pS_0(u-d)h_0$ if the market asset moves up or a decrease of $qS_0(u-d)h_0$ if the market asset moves down. For $p > 0.5$, the depreciation in the premium if the spot price moves down $pS_0(u-d)h_0$ is larger than the appreciation in the premium if the spot price moves upwards $qS_0(u-d)h_0$, i.e.,

$$qS_0(u-d)h_0 < pS_0(u-d)h_0$$

net of cash. This then balances out since upsteps occur more frequently for $p > 0.5$. These can be rearranged into the form

$$
\begin{aligned}
f_u &= (1+r)f_0 + S_0uh_0 - (pu+qd)S_0h_0\\
f_d &= (1+r)f_0 + S_0dh_0 - (pu+qd)S_0h_0
\end{aligned}
$$

Net of the term amortizing the discount factor, $(1+r)f_0$, this amounts to **time decay in (i)** the **deferred spot transaction of the derivative** $-(pu+qd)S_0h_0$ and **(ii)** the **insurance aspect of the option** which is either S_0uh_0 or S_0dh_0 . This is amortization of premium in proportion to the market return and delta for time decay in the economic value of the deferred transaction of the derivative for that binomial step, plus binomial variation dependent upon ΔS_0 because it is an option proper.

With the offsetting deltas, the portfolio change is deterministic provided that the premium is marketable at the bookvalue f_1. A deterministic portfolio return is possible because replicating and offsetting the instantaneous option delta is sufficient for a binomial underlying asset and a binomial asset only.

The partial change in the option premium with respect to the market asset is

$$f_u - f_d = v^{n-1}\sum_{x=a}^{n} (S_0 u^x d^{n-x} - k)\left(\frac{{}^{n-1}C_{x-1}}{p} - \frac{{}^{n-1}C_x}{q}\right) p^x q^{n-x}$$

and the delta is the partial difference quotient

$$h_0 = \frac{f_u - f_d}{S_u - S_d} = v^{n-1}\sum_{x=a}^{n} \frac{S_0 u^x d^{n-x} - k}{S_0(u-d)}\left(\frac{{}^{n-1}C_{x-1}}{p} - \frac{{}^{n-1}C_x}{q}\right) p^x q^{n-x}$$

Note that by rearranging the backward recursion of the premium in the forms $f_u = (1+r) f_0 + qS_0(u-d)h_0$ and $f_d = (1+r) f_0 - pS_0(u-d)h_0$ two alternative expressions for the delta are

$h_0 = \frac{f_u - (1+r) f_0}{qS_0(u-d)}$ in terms of an upstep and $h_0 = \frac{(1+r) f_0 - f_d}{pS_0(u-d)}$ in terms of a downstep.

B. The delta neutral option portfolio

The portfolio value at time zero is

$$\begin{aligned}\Pi_0 &= S_0 h_0 - f_0 \\ &= S_0 \frac{f_u - f_d}{S_u - S_d} - f_0\end{aligned}$$

In one periods time, the value of the portfolio is

$$\Pi_1 = S_1 h_1 - f_1$$

If the portfolio is rebalanced at that time with the mandatory delta update of $S_1(h_1 - h_0)$, then the portfolio change net of the update is partial with respect to the market asset and then comprises terms of return only; the residual statistical variation is born by the rebalance in which case, net of the rebalance, the portfolio value is

$$\Pi_1 = S_1 h_0 - f_1$$

as per the unchanged delta. If the spot price of the asset goes up in value to $S_u = S_0 u$ then the portfolio becomes

$$\begin{aligned}\Pi_u &= S_u h_0 - f_u \\ &= S_u \frac{f_u - f_d}{S_u - S_d} - f_u\end{aligned}$$

that is

$$\Pi_u = S_u \frac{f_u - f_d}{S_u - S_d} - f_u = S_0 u \frac{f_u - f_d}{S_0(u-d)} - f_u = \frac{uf_u - uf_d - (u-d) f_u}{u-d} = \frac{df_u - uf_d}{u-d}$$

If the asset goes down in value to $S_d = S_0 d$ then the portfolio becomes

$$\begin{aligned}\Pi_d &= S_d h_0 - f_d \\ &= S_d \frac{f_u - f_d}{S_u - S_d} - f_d\end{aligned}$$

that is

$$\Pi_d = S_d \frac{f_u - f_d}{S_u - S_d} - f_d = S_0 d \frac{f_u - f_d}{S_0(u-d)} - f_d = \frac{df_u - df_d - (u-d) f_d}{u-d} = \frac{df_u - uf_d}{u-d}$$

In either case, net of the outstanding delta-update due, the value of the portfolio is

$$\Pi_1 = \frac{df_u - uf_d}{u - d}$$

since the position in the binomial asset $h_0 S_0$ has matched and offset the option delta and the binomial variation ΔS, net of the update, since delta is sufficient for the binomial variable. The remaining binomial variation, to complete the total change and keep the instantaneous delta matched, $S_1 \Delta h_0$, is then born by rebalancing and transferred into the delta cost (which is sourced from equity), noting that both the **quantity amount** of the rebalance Δh_0 (since delta is risky for options proper) and the price [value amount] at which the asset is traded S_1 are both stochastic and hence risky; this risk is paid for "ex-post", i.e., "after the fact" since it is incurred as conditional on S_0. It should be noted that paying for risk "ex-post" is not risk free. It is also reminded once more that every time the market moves, delta needs to be rebalanced. Given that the mean return per step of the asset[4] is $pu + qd$

$$\frac{E(S_1 \mid S_0)}{S_0} = \frac{p(S_0 u) + q(S_0 d)}{S_0} = pu + qd$$

the value of the portfolio in terms of the return components net of the delta update is

$$\Pi_1 = (pu + qd) S_0 h_0 - (1 + r) f_0$$

This is not a fixed rate of return at the cash rate since the asset earns more than the cash rate, i.e., it is not $\Pi_1 = (1+r) S_0 h_0 - (1+r) f_0 = (1+r)\Pi_0$ which is now proved in **(C)**. Holding the portfolio return fixed flat at the cash rate debases the portfolio and castrates the asset of return and demeans it into an oxymoron[5] risky bank account Y where

$$\frac{E(Y_1 \mid Y_0)}{Y_0} = 1 + r$$

by demeaning $pu + qd$ by varying either probability p or return factor u to take up the slack between return and no return. If the actual asset S is traded with an option priced for Y, then the resulting mispricing in the debased premium kills the return on the market asset S. For a call, the option is underpriced with the debased premium, and when it changes with respect to the market asset S rather than the demeaned asset Y, it undergoes greater changes to make up ground for the mispricing. A short position in the premium therefore incurs more than the cash rate [in fact, it incurs even more than the market return on S] which then kills the return on the market asset S and demeans it to that which would have been earned if the demeaned asset itself was the security that was traded to replicate and nullify the option at the debased price which is unbiased and accurate only for the demeaned asset Y.

C. Delta neutral portfolios need not earn the cash rate *r* net of delta updates

Proof 1

Heuristically, given that the deltas are matched so that the portfolio value is the same whether the spot price of the asset moves upwards or downwards, the portfolio change is then a variable without statistical variation and the forward value of the variable coincides with its expectation $\Pi_1 = E(\Pi_1 \mid \Pi_0)$ just before the delta is rebalanced which then completes the total portfolio change and sources the residual binomial variation. Given Π_0, which is synonymous with conditioning on the spot price S_0, and given that the option delta is matched with the offsetting asset (and must be rebalanced at the end of the step and rebalanced for every single binomial fluctuation), the portfolio value at the end of the step just before the delta update is

$$\begin{aligned}
\Pi_1 &= E(\Pi_1 \mid \Pi_0) \\
&= p\Pi_u + q\Pi_d \\
&= p(S_u h_0 - f_u) + q(S_d h_0 - f_d) \\
&= p(S_0 u h_0 - f_u) + q(S_0 d h_0 - f_d) \\
&= (pu + qd) S_0 h_0 - (pf_u + qf_d)
\end{aligned}$$

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[4] equivalent in continuous time to $\mu + \frac{1}{2}\sigma^2$ which is the coefficient of the first infinitesimal parameter $(\mu + \frac{1}{2}\sigma^2) S_t \, dt$ of proportional Brownian motion.

[5] **oxymoron**[ox-see-**more**-on] *n* figure of speech that combines two ~~apparently~~ contradictory ideas (Collins Gem, Bank of English)

Substituting the backwards recursion for the option premium accurate for the market asset $vpf_u + vqf_d = f_0$

$$\begin{aligned}\Pi_1 &= (pu+qd)S_0h_0 - (1+r)(vpf_u + vqf_d) \\ &= (pu+qd)S_0h_0 - (1+r)\,f_0\end{aligned}$$

showing that the portfolio earns the market rate of return on the asset *S* and incurs the cash rate on the option as valued by its premium. It is shown shortly that if the portfolio is debased with the constraint $\Delta\Pi_0 = r\Pi_0$, which holds the portfolio return fixed and flat at the cash rate, then backing out the recursion for the premium returns that of the debased premium void of market return rather than that of the statistically accurate premium for the market asset to deliver which is free to earn market return on delta. **The debased premium is therefore a direct result of debasing the portfolio by holding it flat at *r* and killing the market flux with** $\Delta\Pi_0 = r\Pi_0$.

For a variable to be equal to its expectation, the variable's variance must be zero; $Var\left(\Pi_1 \mid \Pi_0\right) = 0$ in this context here. To be rigorous, this is now shown; it is firstly proved that the variance of the delta position for the replicating asset, the option, and the covariance between the two are in parity

$$Var\left(S_1h_0 \mid S_0\right) = Var\left(f_1 \mid S_0\right) = Cov\left(S_1h_0, f_1 \mid S_0\right) = h_0^2S_0^2pq(u-d)^2$$

- variance of offsetting delta position in market asset for an unchanged delta

  $$Var\left(S_1 \mid S_0\right) = S_0^2pq(u-d)^2$$

- variance of option premium

  $$\begin{aligned}Var\left(f_1 \mid S_0\right) &= pf_u{}^2 + qf_d{}^2 - (pf_u + qf_d)^2 \\ &= pf_u{}^2 + qf_d{}^2 - (p^2f_u{}^2 + q^2f_d{}^2 + 2pqf_uf_d) \\ &= pqf_u{}^2 + pqf_d{}^2 - 2pqf_uf_d \\ &= pq(f_u{}^2 + f_d{}^2 - 2f_uf_d) \\ &= pq(f_u - f_d)^2 \\ &= \left((f_u - f_d)^2/(S_0u - S_0d)^2\right)S_0^2pq(u-d)^2 \\ &= h_0^2S_0^2pq(u-d)^2\end{aligned}$$

- covariance between option premium and offsetting delta position in market asset

  $$\begin{aligned}Cov\left(S_1, f_1 \mid S_0\right) &= E\left(S_1f_1 \mid S_0\right) - E\left(S_1 \mid S_0\right)E\left(f_1 \mid S_0\right) \\ &= (pS_uf_u + qS_df_d) - (pS_u + qS_d)(pf_u + qf_d) \\ &= (puS_0f_u + qdS_0f_d) - (puS_0 + qdS_0)(pf_u + qf_d) \\ &= puS_0f_u + qdS_0f_d - (p^2uS_0f_u + pqdS_0f_u + pquS_0f_d + q^2dS_df_d) \\ &= pquS_0f_u + pqdS_0f_d - (pqdS_0f_u + pquS_0f_d) \\ &= pq(u-d)S_0f_u - pq(u-d)S_0f_d \\ &= pq(u-d)S_0(f_u - f_d) \\ &= \left((f_u - f_d)/(S_0u - S_0d)\right)pq(u-d)^2S_0^2 \\ &= h_0S_0^2pq(u-d)^2\end{aligned}$$

showing that the variance of the accurate option is in proportion with the market asset where the constant of proportionality is the square of the option delta. Since the replicating asset is held in proportion to the option delta, the variance of the position in the replicating asset matches that of the option in order to offset it as the point of the replication. This kills the option and its variance and nets out the second cumulant of the basis leaving the first cumulant for the return on the market asset *S* and a convex portfolio. This return profile is **(1)** the economic value of the deferred transaction of the derivative and **(2)** the value of the financial service of making

a market in the derivative and taking up the pecuniary interest in the deferred transaction. The variance of the portfolio for an unchanged delta is then zero [net of the update[6]]

$$\begin{aligned} Var\left(\Pi_1 \mid \Pi_0\right) &= Var\left(S_1 h_0 - f_1 \mid S_0\right) \\ &= Var\left(S_1 h_0 \mid S_0\right) + Var\left(f_1 \mid S_0\right) - 2Cov\left(S_1 h_0, f_1 \mid S_0\right) \\ &= h_0^2 Var\left(S_1 \mid S_0\right) + Var\left(f_1 \mid S_0\right) - 2h_0 Cov\left(S_1, f_1 \mid S_0\right) \\ &= h_0^2 S_0^2 pq(u-d)^2 + h_0^2 S_0^2 pq(u-d)^2 - 2h_0^2 S_0^2 pq(u-d)^2 \\ &= 0 \end{aligned}$$

as required, so that indeed $\Pi_1 = E\left(\Pi_1 \mid \Pi_0\right)$ and $\Pi_1 = E\left(\Pi_1 \mid \Pi_0\right) = (pu+qd)S_0 h_0 - (1+r) f_0$ proved rigorously.

Proof 2

From the backwards recursion of the option premium accurate for the market asset *S*

$$f_0 = vpf_u + vqf_d \qquad (1+r) f_0 = pf_u + qf_d$$

- Putting $(1+r) f_0 = pf_u + qf_d$ in terms of the forward option premium for an upstep

  $(1+r) f_0 = pf_u + q(f_d - f_u + f_u) = pf_u + qf_u - q(f_u - f_d) = f_u - qS_0(u-d)h_0$

  so that $f_u = (1+r) f_0 + qS_0(u-d)h_0$

- Putting $(1+r) f_0 = pf_u + qf_d$ in terms of the forward option premium for a downstep

  $(1+r) f_0 = p(f_u - f_d + f_d) + qf_d = p(f_u - f_d) + pf_d + qf_d = pS_0(u-d)h_0 + f_d$

  so that $f_d = (1+r) f_0 - pS_0(u-d)h_0$

Hence,

(1) $f_u = (1+r) f_0 + qS_0(u-d)h_0$
(2) $f_d = (1+r) f_0 - pS_0(u-d)h_0$

If the market asset moves up, the portfolio becomes

$$\begin{aligned} \Pi_u &= S_u h_0 - f_u \\ &= S_0 u h_0 - \left((1+r) f_0 + qS_0(u-d)h_0\right) \\ &= S_0(u - qu + qd)h_0 - (1+r) f_0 \\ &= (pu+qd)S_0 h_0 - (1+r) f_0 \end{aligned}$$

If the market asset moves down, then the portfolio becomes

$$\begin{aligned} \Pi_d &= S_d h_0 - f_d \\ &= S_0 d h_0 - \left((1+r) f_0 - pS_0(u-d)h_0\right) \\ &= S_0(d + pu - pd)h_0 - (1+r) f_0 \\ &= (pu+qd)S_0 h_0 - (1+r) f_0 \end{aligned}$$

That is, in both cases, the portfolio earns the market rate of return on the market asset *S* and incurs the cash rate on the short option

$$\Pi_1 = (pu+qd)S_0 h_0 - (1+r) f_0$$

net of the mandatory delta rebalance outstanding for the step to complete the total portfolio change. If the change in *S* is factored into the change in the option premium, then

(1) $f_u = (1+r) f_0 + S_0 u h_0 - (pu+qd)S_0 h_0$
(2) $f_d = (1+r) f_0 + S_0 d h_0 - (pu+qd)S_0 h_0$

[6] for which the variance is $Var\left(S_1(h_1 - h_0) \mid S_0\right) = S_0{}^2 pq\left(u(h_u - h_0) - d(h_d - h_0)\right)^2$

so that

$$\Pi_u = S_u h_0 - f_u = S_0 u h_0 - \left((1+r) f_0 + S_0 u h_0 - (pu+qd) S_0 h_0\right) = (pu+qd) S_0 h_0 - (1+r) f_0$$
$$\Pi_d = S_d h_0 - f_d = S_0 d h_0 - \left((1+r) f_0 + S_0 d h_0 - (pu+qd) S_0 h_0\right) = (pu+qd) S_0 h_0 - (1+r) f_0$$

as before.

Proof 3

For further rigorous mathematics, this is now proved combinatorially. Noting, as proved for the backwards recursion in the option premium for f_u and f_d, that

(1) $df_u = \frac{d}{p} v^{n-1} \sum_{x=a}^{n} (S_0 u^x d^{n-x} - k)\, {}^{n-1}C_{x-1} p^x q^{n-x}$ **(2)** $uf_d = \frac{u}{q} v^{n-1} \sum_{x=a}^{n} (S_0 u^x d^{n-x} - k)\, {}^{n-1}C_x p^x q^{n-x}$

so

$$df_u - uf_d = v^{n-1} \sum_{x=a}^{n} (S_0 u^x d^{n-x} - k) \left(\frac{d}{p} {}^{n-1}C_{x-1} - \frac{u}{q} {}^{n-1}C_x \right) p^x q^{n-x}$$

Further noting that

$$\begin{aligned}
\frac{d}{p} {}^{n-1}C_{x-1} - \frac{u}{q} {}^{n-1}C_x &= \frac{qd\, {}^{n-1}C_{x-1} - pu\, {}^{n-1}C_x}{pq} \\
&= \frac{q(p+q) d\, {}^{n-1}C_{x-1} - p(p+q) u\, {}^{n-1}C_x}{pq} \\
&= \frac{pqd\, {}^{n-1}C_{x-1} + q^2 d\, {}^{n-1}C_{x-1} - p^2 u\, {}^{n-1}C_x - pqu\, {}^{n-1}C_x}{pq} \\
&= \frac{q^2 d\, {}^{n-1}C_{x-1} - pqu\, {}^{n-1}C_x}{pq} - \frac{p^2 u\, {}^{n-1}C_x - pqd\, {}^{n-1}C_{x-1}}{pq} \\
&= \frac{qd\, {}^{n-1}C_{x-1} - pu\, {}^{n-1}C_x}{p} - \frac{pu\, {}^{n-1}C_x - qd\, {}^{n-1}C_{x-1}}{q}
\end{aligned}$$

and further, since ${}^{n-1}C_{x-1} + {}^{n-1}C_x = {}^{n}C_x$, that

$$\begin{aligned}
\frac{d}{p} {}^{n-1}C_{x-1} - \frac{u}{q} {}^{n-1}C_x &= \frac{qd\, {}^{n-1}C_{x-1} - pu({}^{n}C_x - {}^{n-1}C_{x-1})}{p} - \frac{pu\, {}^{n-1}C_x - qd({}^{n}C_x - {}^{n-1}C_x)}{q} \\
&= \frac{(pu+qd)\, {}^{n-1}C_{x-1} - pu\, {}^{n}C_x}{p} - \frac{(pu+qd)\, {}^{n-1}C_x - qd\, {}^{n}C_x}{q} \\
&= \left(\frac{(pu+qd)\, {}^{n-1}C_{x-1}}{p} - u\, {}^{n}C_x \right) - \left(\frac{(pu+qd)\, {}^{n-1}C_x}{q} - d\, {}^{n}C_x \right)
\end{aligned}$$

Hence

$$\begin{aligned}
df_u - uf_d &= v^{n-1} \sum_{x=a}^{n} (S_0 u^x d^{n-x} - k) \left\{ \left(\frac{(pu+qd)\, {}^{n-1}C_{x-1}}{p} - u\, {}^{n}C_x \right) - \left(\frac{(pu+qd)\, {}^{n-1}C_x}{q} - d\, {}^{n}C_x \right) \right\} p^x q^{n-x} \\
&= v^{n-1} \sum_{x=a}^{n} (S_0 u^x d^{n-x} - k) \left(\frac{(pu+qd)\, {}^{n-1}C_{x-1}}{p} - \frac{(pu+qd)\, {}^{n-1}C_x}{q} - (u-d)\, {}^{n}C_x \right) p^x q^{n-x} \\
&= (pu+qd) v^{n-1} \sum_{x=a}^{n} (S_0 u^x d^{n-x} - k) \left(\frac{{}^{n-1}C_{x-1}}{p} - \frac{{}^{n-1}C_x}{q} \right) p^x q^{n-x} \\
&\qquad - (u-d) v^{n-1} \sum_{x=a}^{n} (S_0 u^x d^{n-x} - k)\, {}^{n}C_x p^x q^{n-x} \\
&= (pu+qd)\left(f_u - f_d \right) - (u-d)(1+r) f_0
\end{aligned}$$

So

$$\begin{aligned}\Pi_1 &= \frac{df_u - uf_d}{u-d} \\ &= \frac{(pu+qd)\left(f_u - f_d\right)-(u-d)(1+r)f_0}{u-d} \\ &= \frac{(pu+qd)S_0(f_u - f_d)}{S_0(u-d)} - (1+r)f_0 \\ &= (pu+qd)S_0\frac{f_u - f_d}{S_u - S_d} - (1+r)f_0\end{aligned}$$

that is,

$$\Pi_1 = (pu+qd)S_0h_0 - (1+r)f_0$$

which is the market return on the market asset and the cash rate on the short option.

The proof is complete. Quod Erat Demonstrandum, as was to be demonstrated.

The proof for a put option is similar and the result is identical. I have hence proved that the return on the portfolio for the traded market asset and the statistically unbiased premium is equal to **(1)** the return on the delta position in the market asset at the rate of return $pu + qd - 1$ on the market asset S and **(2)** the cash rate r on the short position in the option, i.e.,

$$\Pi_1 = (pu+qd)S_0h_0 - (1+r)f_0$$

$$\Pi_1 - \Pi_0 = (pu+qd-1)S_0h_0 - rf_0 \qquad \Delta\Pi_0 = (pu+qd-1)S_0h_0 - rf_0 > r\Pi_0$$

$$\frac{\Pi_1 - \Pi_0}{\Pi_0} = (pu+qd-1)\frac{S_0h_0}{\Pi_0} - r\frac{f_0}{\Pi_0} \qquad \frac{\Delta\Pi_0}{\Pi_0} = (pu+qd-1)\frac{S_0h_0}{\Pi_0} - r\frac{f_0}{\Pi_0} > r$$

which is not fixed at the cash rate, nor flat from step to step due to a changing composition of the portfolio for every variable rebalance to re-neutralize delta after each market fluctuation to keep the delta position replicating and matched to the option delta. Together with the delta update, the portfolio value at time 1 is

$$\Pi_1 = (pu+qd)S_0h_0 - (1+r)f_0 + S_1\Delta h_0$$

with a total change in the delta-neutral portfolio of

$$\begin{aligned}\Pi_1 - \Pi_0 &= (pu+qd)S_0h_0 - (1+r)f_0 + S_1(h_1 - h_0) - \left(S_0h_0 - f_0\right) \\ &= (pu+qd-1)S_0h_0 - rf_0 + S_1(h_1 - h_0)\end{aligned}$$

that is

$$\Delta\Pi_0 = (pu+qd-1)S_0h_0 - rf_0 + S_1\Delta h_0$$

This is now proved in a discrete calculus setting with the backward recurrence of the walk, in parallel to the partial difference equations in continuous time for which combinatorial methods are not available to do the proofs. As is shown shortly, if one then backs out the partial difference (discrete) or differential (continuous) equation from the return components, one obtains the correct and accurate premium for the market asset to deliver. If one instead backs out the equation for the premium from the debased portfolio held fixed and flat at the cash rate, then one ends up with the PDE of the debased premium of Black and Scholes (1973) that misprices and 'systematically deviates' from the option for the actual asset to deliver *S*. The backed out PDE from the correct return components is that of the unbiased accurate premium inclusive of market return, not the PDE of a statistically biased and inaccurate debased option premium void of market return with 100% of the market risk. The debased price is hence return-free, not risk-free. This is the reason for the skew in implied volatility and 'systematic deviation' in the debased price consistently observed since 1973 whereby slack Δ is introduced via the first cumulant to demean the drift by debasing the portfolio and premium, and then backed out again via the second cumulant in volatility. The proof for the portfolio return components within the context of discrete calculus and partial difference equations is considered now.

The Return Components of a Binomial Replicating Portfolio of the Statistically Unbiased Option for the Market Asset S: Proof within a discrete calculus setting

The mathematics of the return components of the option replicating portfolio is now composed into a calculus format, the discrete calculus and difference equations. Specifically, the backwards deconvolution recurrence equation of the option premium for the proportional binomial market asset *S* is derived and then utilized to deduce the return components of the total change in the replicating portfolio net of the delta rebalance. This is in parallel, and to support, the mathematics of continuous time. The methods composed in a format consistent with the calculus provide a rigorous alternative to combinatorial methods and provide for a full calculus of the dynamics of the walk and its derived functions. Obtaining the backward deconvolution recurrence of the premium is split and built up in stages (**A**, **B**, **C**, **D**, **E**, and **F**) and the return components are deduced with several further stages (**G** and **H**); this parallels the continuous time results for proportional Brownian motion almost one-to-one [**which is intentional since I have tried to make them as consistent as possible**].

The variable X_n is the step generator of any random walk for an n step transition, and $X_n \mid X_j$ is the generator for the walk for the transition interval between (and including) the j^{th} step and the n^{th} step. The conditional variable is binomial distributed, $X_n \mid X_j \sim B(n-j, p)$ and the discrete transition probability function is $p_{X_n|X_j}(x \mid x_0) = {}^{n-n_0}C_{x-x_0}\, p^{x-x_0} q^{n-j-(x-x_0)}$. The variables n and x are the forward variables and j and x_0 are the backward variables of the discrete diffusion variable of the step generator which is a cumulative Bernoulli basis. The combinatorial relation of **Pascal's triangle** forms the **backbone** from which all of the binomial dynamics derive, and interestingly, when each row of Pascal's triangle is scaled by 2^n, forms a **Brownian motion transition surface** and is an expression of the **heat equation** and the **Central Limit Theorem**. Consequently, the key relationship of the step generator is Pascal's triangle and the backwards binomial recursion for which forms the **backbone** of the backward equation [as well as the forward equation] describing how the initial steps of the cumulative walk are deconvolved from the beginning of the cumulative total of aggregated steps. After scaling the components of the step generator (from 0 and 1 to $\log u$ and $\log d$), this forms the basis for the exponential (proportional) walk. The equation required to calculate the between node return for the replicating portfolio is then the backward recurrence of the option premium. This is the discrete time discrete diffusion (discrete basis function) analogue to the conjugate heat equation on logarithms and is a recurrence for partial changes in the two independent variables (similarly to the heat equation), which are the backward variables j and S_j where j is the backward time variable and S_j is the backward walk variable. These are the two independent variables also of the delta h and the hedge portfolio Π. Additionally, the forward variables of S are required explicitly (since the premium is summed over y) which are n and y, n being the forward time variable of S and y being the forward walk variable and distributional argument of the PMF of S. Hence, the step generator is then transformed to that of the proportional random walk and then to that of the premium whereby each pair of forward and backward variables of the step generator transform to those of the exponential (proportional) walk and premium. The option expires when the backward time variable reaches the forward time variable $j = n$ which is in $n - j$ steps time and a forward transition of $n - j$ steps in the walk variable. This forms the boundary condition $f_{n,n}$ to the partial difference equation, as for Brownian motion in continuous time, for which the initial condition $f_{0,n}$ is the premium of interest and for which $f_{j,n}$ is replicated as a bookvalue by $S_j\, h_{j,n}$ within the replicating portfolio.

Discrete calculus proof that a delta-neutralized portfolio for a statistically accurate premium earns the market return on the market asset and incurs the cash rate on the option

The intermediate steps and formulae are listed, consisting of eight steps A through to I. The mathematics of the [derivation of the] discrete time discrete diffusion analogues inclusive of those of delta [and the mathematics of unfolding it, and techniques of debasing it and separating out the mispricing] to the conjugate heat equation are original to this thesis.

With the foundation theory, the backwards deconvolution recurrence equation of the option premium is derived starting with **(A)** and **(B)** by obtaining a backward recurrence of the probability function of the step generator X, which is the PMF of a standard binomial random variable and cumulative binomial basis, then **(C)** and **(D)** transforms this recurrence to the backward recurrence equation of the PMF of the multiplicative random walk S, **(E)** transforms this to that of the discounted density and then finally **(F)** transforms the equation to that of the option premium. This is a discrete version of the method that is applied to the continuous time equivalent, that of proportional Brownian motion, to obtain the partial differential equation for the option premium which is the conjugate heat equation on logarithms with a source for time decay in the cash rate discount factor. **(G)** obtains an expression for the discrete total derivative of the option premium and then **(H)** takes the total derivative of the option replicating portfolio of the replicating asset and the short premium and then substitutes the total de-

rivative of the option premium [original to this thesis] into the total derivative of the option portfolio to deduce the return components of the portfolio change, completing the task and objective of this section[7]; proving within a discrete calculus setting that the portfolio with the statistically accurate premium and its replicating delta is not fixed flat from period to period at the cash rate but instead an interaction between the market return on the market asset and the cash rate on the short option net of the mandatory delta rebalance.

A. Backwards deconvolution equation for the PMF of the cumulative binomial basis *X*

$$p_{X_n|X_j}(x\,|\,x_0) = p \times p_{X_n|X_{j+1}}(x\,|\,x_0+1) + q \times p_{X_n|X_{j+1}}(x\,|\,x_0) \qquad \text{where} \quad p_{X_n|X_j}(x\,|\,x_0) = {}^{n-j}C_{x-x_0}\, p^{x-x_0} q^{(n-j)-(x-x_0)}$$

B. Backwards equation for the PMF of the cumulative binomial basis *X*

$$-\frac{\delta p_{X_n|X_j}(x\,|\,x_0)}{\delta j} = p\,\frac{\delta p_{X_n|X_{j+1}}(x\,|\,x_0)}{\delta x_0} \qquad \text{where} \quad p_{X_n|X_j}(x\,|\,x_0) = {}^{n-j}C_{x-x_0}\, p^{x-x_0} q^{n-j-(x-x_0)}$$

C. Transformation of the step generator *X* to the proportional random walk *S*

$$S_n = s_0 u^{(X_n - x_0)} d^{(n-j)-(X_n-x_0)} \qquad \text{and} \qquad p_{S_n|S_j}(y\,|\,s_0) = p_{X_n|X_j}(x\,|\,x_0)$$

for $$p_{S_n|S_j}(y\,|\,s_j) = {}^{n}C_{\log(y/s_j d^n)/\log(u/d)}\, p^{\log(y/s_j d^n)/\log(u/d)} q^{\log(s_j u^n/y)/\log(u/d)}$$

D. Backwards deconvolution equation for the PMF of the proportional random walk *S*

$$p_{S_n|S_j}(y\,|\,s_0) = p \times p_{S_n|S_{j+1}}(y\,|\,s_0 u) + q \times p_{S_n|S_{j+1}}(y\,|\,s_0 d) \qquad \text{analogue to} \qquad f_t = -(\mu + \tfrac{1}{2}\sigma^2)\, S\, f_S - \tfrac{1}{2}\sigma^2 S^2 f_{SS}$$

where $$p_{S_n|S_j}(y\,|\,s_j) = {}^{n}C_{\log(y/s_j d^n)/\log(u/d)}\, p^{\log(y/s_j d^n)/\log(u/d)} q^{\log(s_j u^n/y)/\log(u/d)}$$

E. Backwards deconvolution equation of the discounted PMF of *S*

$$v^{n-j} p_{S_n|S_j}(y\,|\,s_0) = vp \times v^{n-j-1} p_{S_n|S_{j+1}}(y\,|\,s_0 u) + vq \times v^{n-j-1} p_{S_n|S_{j+1}}(y\,|\,s_0 d) \qquad f_t = r f - (\mu + \tfrac{1}{2}\sigma^2)\, S\, f_S - \tfrac{1}{2}\sigma^2 S^2 f_{SS}$$

F. Backwards deconvolution equation for the premium *f* for the option on *S*

$$f_{j,n}(k\,|\,s_0) = vp \times f_{j+1,n}(k\,|\,s_0 u) + vq \times f_{j+1,n}(k\,|\,s_0 d) \qquad V_t = rV - (\mu + \tfrac{1}{2}\sigma^2)\, S\, V_S - \tfrac{1}{2}\sigma^2 S^2 V_{SS}$$

G. The discrete total derivative of the option premium *f*

$$\frac{\Delta f_{j,n}}{\Delta j} = r f_{j,n} - (pu + qd - 1) S_j h_{j,n} + h_{j,n}\frac{\Delta S_j}{\Delta j} \qquad \text{where} \qquad \frac{\Delta S_j}{\Delta j} = \begin{cases} S_j(u-1) \text{ with probability } p \\ S_j(d-1) \text{ with probability } q \end{cases}$$

H. The total derivative of the option portfolio Π and return components of ΔΠ

$$\frac{\Delta \Pi_{j,n}}{\Delta j} = S_j \frac{\Delta h_{j,n}}{\Delta j} + h_{j,n}\frac{\Delta S_j}{\Delta j} - \frac{\Delta f_{j,n}}{\Delta j} \qquad \text{return components} \quad \frac{\Delta \Pi_{j,n}}{\Delta j} = (pu + qd - 1) S_j h_{j,n} - r f_{j,n} + S_{j+1}\frac{\Delta h_{j,n}}{\Delta j}$$

If **(i)** $u = 1/d$, so that the walk is symmetric on logarithms $\log u = -\log d$, and **(ii)** k is on the sample space of the forward variables (or the premium is for a future or a pseudo option), then the backwards recursion in the premium can be put into a format with partial derivatives

$$\frac{\delta f_{j,n}}{\delta j} = r f_{j,n} - (pu + qd - 1)\, s_j h_{j,n}$$

A. Backwards deconvolution equation for the PMF of the cumulative binomial basis *X*

A cumulative Bernoulli basis function generates the dynamics of any binomial walk over any forward transition interval of given length. The PMF of the cumulative basis is

$$p_{X_n|X_j}(x\,|\,x_0) = {}^{n-j}C_{x-x_0}\, p^{x-x_0} q^{(n-j)-(x-x_0)}$$

[7] this method and the results [among other things, the continuous time results] were presented at the 2010 University of Sydney Finance PhD Colloquium.

inclusive of backward variables. The backwards recursion of this PMF is now derived based upon Pascal's triangle for the recursion. From Pascal's triangle ${}^{n+1}C_r = {}^{n}C_r + {}^{n}C_{r-1}$ i.e., ${}^{n}C_r = {}^{n+1}C_r - {}^{n}C_{r-1}$, so that

$$\begin{aligned}
&p_{X_n|X_{j+1}}(x\,|\,x_0) - p_{X_n|X_j}(x\,|\,x_0) \\
&\quad = {}^{n-(j+1)}C_{x-x_0}\, p^{x-x_0} q^{n-(j+1)-(x-x_0)} - {}^{n-j}C_{x-x_0}\, p^{x-x_0} q^{n-j-(x-x_0)} \\
&\quad = \left({}^{n-j}C_{x-x_0} - {}^{n-(j+1)}C_{x-x_0-1}\right) p^{x-x_0} q^{n-(j+1)-(x-x_0)} - {}^{n-j}C_{x-x_0}\, p^{x-x_0} q^{n-j-(x-x_0)} \\
&\quad = \frac{1}{q}\,{}^{n-j}C_{x-x_0}\, p^{x-x_0} q^{(n-j)-(x-x_0)} - \frac{p}{q}\,{}^{n-(j+1)}C_{x-x_0-1} p^{x-x_0-1} q^{n-(j+1)-(x-x_0-1)} - {}^{n-j}C_{x-x_0}\, p^{x-x_0} q^{n-j-(x-x_0)} \\
&\quad = \frac{p}{q}\,{}^{n-j}C_{x-x_0}\, p^{x-x_0} q^{(n-j)-(x-x_0)} - \frac{p}{q}\,{}^{n-(j+1)}C_{x-(x_0+1)}\, p^{x-(x_0+1)} q^{n-(j+1)-[x-(x_0+1)]} \\
&\quad = \frac{p}{q}\left(p_{X_n|X_j}(x\,|\,x_0) - p_{X_n|X_{j+1}}(x\,|\,x_0+1)\right)
\end{aligned}$$

Multiplying both sides by the probability of a downstep q

$$q \times \left(p_{X_n|X_{j+1}}(x\,|\,x_0) - p_{X_n|X_j}(x\,|\,x_0)\right) = p \times \left(p_{X_n|X_j}(x\,|\,x_0) - p_{X_n|X_{j+1}}(x\,|\,x_0+1)\right)$$

and shifting to the same side the probability functions relating to a transition interval of a given length and common number of steps in the cumulative walk from the backward variable to the forward variable

$$p_{X_n|X_j}(x\,|\,x_0) = p \times p_{X_n|X_{j+1}}(x\,|\,x_0+1) + q \times p_{X_n|X_{j+1}}(x\,|\,x_0)$$

This can be deduced by purely logical reasoning as per the COMPENDIUM.

B. Backwards equation for the PMF of the cumulative binomial basis *X*

From the backwards recurrence relation,

$$\begin{aligned}
p_{X_n|X_j}(x\,|\,x_0) &= p \times p_{X_n|X_{j+1}}(x\,|\,x_0+1) + q \times p_{X_n|X_{j+1}}(x\,|\,x_0) \\
&= p \times p_{X_n|X_{j+1}}(x\,|\,x_0+1) + (1-p) \times p_{X_n|X_{j+1}}(x\,|\,x_0)
\end{aligned}$$

or

$$p_{X_n|X_j}(x\,|\,x_0) - p_{X_n|X_{j+1}}(x\,|\,x_0) = p \times \left(p_{X_n|X_{j+1}}(x\,|\,x_0+1) - p_{X_n|X_{j+1}}(x\,|\,x_0)\right)$$

which when described in terms of partial differences with unit increment span ($\delta / \delta j$ acting as an operator)

$$\frac{\delta p_{X_n|X_j}(x\,|\,x_0)}{\delta j} = -p\,\frac{\delta p_{X_n|X_{j+1}}(x\,|\,x_0)}{\delta x_0}$$

which is the PDE for $p_{X_n|X_j}(x\,|\,x_0) = {}^{n-j}C_{x-x_0}\, p^{x-x_0} q^{n-j-(x-x_0)}$. This is the backward equation for a process characterized by a unit step function with geometric timing of each unit sultus, obtained directly from the backwards binomial recursion from Pascal's triangle. Note that by simply converting the discrete differences to continuous differentials, the PMF would just be a unit trace of linear drift rather than a normal transition surface, i.e., simply the transition PMF of linear drift with negative slope p for time decay with forward movements in the backward variables. This is because it is high SNR (signal to noise [ratio]) until the Bernoulli basis components are scaled so that it is not simply a unit step function at each sultus. As it stands, prior to scaling the basis components, this is the discrete time analogue to the Poisson process with geometric times between unit increases rather than exponential times. The backward equation of the Poisson process is that with the δ on the time side converted to d since it is continuous in the time variable and discrete in the diffusion variable due to the discrete basis function (and hence also in δ on the diffusion side) as per the discussion in the COMPENDIUM OF RANDOM WALKS. The backward equation is also a unit trace when the δ on the diffusion side goes to d and this is seen by simulating it; the effect of $\delta \to d$ occurs as the number of steps become large.

parameter of step generator	
p	0.8

sample space

Cumulative Bernoulli Basis, X

0	1	2	3	4
				4
			3	
		2		3
	1		2	
0		1		2
	0		1	
		0		1
			0	
				0

distribution

PMF

0	1	2	3	4
				0.4096
			0.512	
		0.64		0.4096
	0.8		0.384	
1		0.32		0.1536
	0.2		0.096	
		0.04		0.0256
			0.008	
				0.0016

C. Transformation of the step generator *X* to the proportional random walk *S*

The step generator conditional on the backward variables x_0 and j of the transition interval $n-j$ is $X_n \mid X_j = x_0$, and the transformation of this variable to that of the proportional binomial random walk $S_n \mid S_j = s_0$ is

$$S_n = s_0 u^{(X_n - x_0)} d^{(n-j)-(X_n - x_0)}$$

conditional on $S_j = s_0$ noting that $S_n = s_0 \exp\big((n-j)d + (X_n - x_0)\log(u/d)\big)$ which is an unmoderated exponential diffusion since X_n is a unit root additive diffusion. This is relevant because debased premiums assume that there are exponential market assets with no risk that can source systematically mispriced options, which is not sensible given what is known about the relationship between risk and return. A Jacobian scale is required only when a basis function is continuous. Since the basis function of the binomial walk is discrete, the probability distribution remains unchanged from p_X to p_S and the transformation means only a variation in the sample space for the forward transition variable from $x = x_0, x_0+1, x_0+2, x_0+3, ..., x_0+n-1, x_0+n$ to

$$y = S_0 d^n,\ S_0 u^1 d^{n-1}, S_0 u^2 d^{n-2},\ S_0 u^3 d^{n-3} ...,\ S_0 u^{n-2} d^2,\ S_0 u^{n-1} d^1,\ S_0 u^n$$

Hence,

$$p_{S_n|S_j}(y \mid s_0) = p_{X_n|X_j}(x \mid x_0)$$

so that

$$p_{S_n|S_j}(y \mid s_j) = {}^nC_{\log(y/s_j d^n)/\log(u/d)}\, p^{\log(y/s_j d^n)/\log(u/d)} q^{\log(s_j u^n/y)/\log(u/d)}$$

for $y = S_0 d^n,\ S_0 u^1 d^{n-1}, S_0 u^2 d^{n-2},\ S_0 u^3 d^{n-3} ...,\ S_0 u^{n-2} d^2,\ S_0 u^{n-1} d^1,\ S_0 u^n$

D. Backwards deconvolution equation for the PMF of the proportional random walk S

If the spot price of the market asset increases, which is indicated by backward conditioning at $X_{j+1} = x_0 + 1$, then $S_{j+1} = s_0 u$. If the spot decreases, which is indicated by no change in conditioning at the backward variables $X_{j+1} = x_0$, then $S_{j+1} = s_0 d$. From the recurrence of the PMF of the walk of the Bernoulli process of the step generator

$$p_{X_n|X_j}(x \mid x_0) = q \times p_{X_n|X_{j+1}}(x \mid x_0) + p \times p_{X_n|X_{j+1}}(x \mid x_0 + 1)$$

the recurrence of the PMF of the proportional binomial walk is then

$$p_{S_n|S_j}(y \mid s_0) = p \times p_{S_n|S_{j+1}}(y \mid s_0 u) + q \times p_{S_n|S_{j+1}}(y \mid s_0 d)$$

where $p_{S_n|S_j}(y \mid s_j) = {}^nC_{\log(y/s_j d^n)/\log(u/d)}\, p^{\log(y/s_j d^n)/\log(u/d)} q^{\log(s_j u^n/y)/\log(u/d)}$. This recursion in the backward walk variable is the backward deconvolution equation of the proportional binomial walk and the discrete two-valued basis function equivalent to the PDE of the transition density for a proportional Brownian motion $f_t = -(\mu + \frac{1}{2}\sigma^2) S f_S - \frac{1}{2}\sigma^2 S^2 f_{SS}$ which is the conjugate heat equation on logarithms and the limit of the backwards binomial recursion as proved in the COMPENDIUM. This recursion is that for the same probability distribution as the step generator *X* but with a changed sample space, as per a transformed dependent variable. This can be seen clearly, when compared to that for the step generator, in the example lattices below for a four step walk;

walk parameters			
s0	110	u	1.12
		d	0.892857
		p	0.8

sample space

Proportional Random Walk, S

0	1	2	3	4
				173.0871
			154.5421	
		137.984		137.984
	123.2		123.2	
110		110		110
	98.21429		98.21429	
		87.69133		87.69133
			78.29583	
				69.90699

distribution

PMF

0	1	2	3	4
				0.4096
			0.512	
		0.64		0.4096
	0.8		0.384	
1		0.32		0.1536
	0.2		0.096	
		0.04		0.0256
			0.008	
				0.0016

E. Backwards deconvolution equation of the discounted PMF of S

Multiplying both sides of the recursion of the PMF of the proportional walk to that which is discounted in preparation to summing for the premium

$$v^{n-j}\,p_{S_n|S_j}(y\,|\,s_0) = p\times v^{n-j}\,p_{S_n|S_{j+1}}(y\,|\,s_0u) + q\times v^{n-j}\,p_{S_n|S_{j+1}}(y\,|\,s_0d)$$

so that

$$v^{n-j}\,p_{S_n|S_j}(y\,|\,s_0) = vp\times\left(v^{n-j-1}\,p_{S_n|S_{j+1}}(y\,|\,s_0u)\right) + vq\times\left(v^{n-j-1}\,p_{S_n|S_{j+1}}(y\,|\,s_0d)\right)$$

This has exactly the same form as that of the option premium, as well as that in the continuous time limit $f_t = r\,f - (\mu + \frac{1}{2}\sigma^2)\,S\,f_S - \frac{1}{2}\sigma^2 S^2 f_{SS}$ which is the conjugate heat equation with a source for time decay in *v*.

F. Backwards deconvolution equation of the premium *f* for the option on S

The statistically accurate option premium is a weighted partial sum of the discounted density and a weighted summation of the discounted PMF. The backwards deconvolution recursion for the premium is now obtained by transforming the backward recurrence of the discounted PMF by weighting it and summing. This is straight forward and gives guidance to what is happening in continuous time. Multiplying the backwards recursion by the moneyness of the option at the forward variables for the option expiry; for a call option, the difference between the forward variable *y* and the strike *k*

$$v^{n-j}(y-k)\,p_{S_n|S_j}(y\,|\,s_0) = vp\times v^{n-j-1}(y-k)\,p_{S_n|S_{j+1}}(y\,|\,s_0u) + vq\times v^{n-j-1}(y-k)\,p_{S_n|S_{j+1}}(y\,|\,s_0d)$$

where *k* is a constant and *y* is the forward walk variable. Summing both sides with respect to the forward walk variable *y* over the region where this is positive $y \geq k$ and therefore inthemoney at the forward variables

$$\begin{aligned} v^{n-j}\sum_{y=a}^{n}(y-k)\,p_{S_n|S_j}(y\,|\,s_0) &= vp\times v^{n-j-1}\sum_{y=a}^{n}(y-k)\,p_{S_n|S_{j+1}}(y\,|\,s_0u) + vq\times v^{n-j-1}\sum_{y=a}^{n}(y-k)\,p_{S_n|S_{j+1}}(y\,|\,s_0d) \\ &= vp\times v^{n-j-1}\sum_{y=a}^{n-1}(y-k)\,p_{S_n|S_{j+1}}(y\,|\,s_0u) + vq\times v^{n-j-1}\sum_{y=a}^{n-1}(y-k)\,p_{S_n|S_{j+1}}(y\,|\,s_0d) \end{aligned}$$

Similarly, for a put,

$$v^{n-j}(k-y)\,p_{S_n|S_j}(y\,|\,s_0) = vp\times v^{n-j-1}(k-y)\,p_{S_n|S_{j+1}}(y\,|\,s_0u) + vq\times v^{n-j-1}(k-y)\,p_{S_n|S_{j+1}}(y\,|\,s_0d)$$

and summing

$$v^{n-j}\sum_{y=0}^{a}(k-y)\,p_{S_n|S_j}(y\,|\,s_0) = vp\times v^{n-j-1}\sum_{y=0}^{a}(k-y)\,p_{S_n|S_{j+1}}(y\,|\,s_0u) + vq\times v^{n-j-1}\sum_{y=0}^{a}(k-y)\,p_{S_n|S_{j+1}}(y\,|\,s_0d)$$

The option premiums are defined as

$$f_{j,n}(k\,|\,s_0) = \begin{cases} v^{n-j}\displaystyle\sum_{y\geq k}(y-k)\,p_{S_n|S_j}(y\,|\,s_0) \\ v^{n-j}\displaystyle\sum_{y\leq k}(k-y)\,p_{S_n|S_j}(y\,|\,s_0) \end{cases} = \begin{cases} \displaystyle\sum_{y=a}^{n}(y-k)v^{n-j}\,p_{S_n|S_j}(y\,|\,s_0) & \text{for a call option} \\ \displaystyle\sum_{y=0}^{a}(k-y)v^{n-j}\,p_{S_n|S_j}(y\,|\,s_0) & \text{for a put option} \end{cases}$$

where *y* is the forward variable of the walk. The backwards recurrence, for either a put or call option, can whence be written as

$$f_{j,n}(k\,|\,s_0) = vp\times f_{j+1,n}(k\,|\,s_0u) + vq\times f_{j+1,n}(k\,|\,s_0d)$$

as obtained by combinatorial methods applied directly to the definition of the premium as a discounted limited expectation of a function g of the cumulative basis variable X. This recurrence describes the partial backwards deconvolution dynamics in terms of variations in the two backward variables j and s_0. The solution to this recurrence is

$$f_{j,n}(k \mid S_j) = \begin{cases} S_j(vpu+vqd)^{n-j}\,\bar{F}_{B(n-j,p')}(u_k) - kv^{n-j}\,\bar{F}_{B(n-j,p)}(u_k) & \text{if it is a call option to buy} \\ kv^{n-j}F_{B(n-j,p)}(u_k) - S_j(vpu+vqd)^{n-j}\,F_{B(n-j,p')}(u_k) & \text{if it is a put option to sell} \end{cases}$$

for the boundary conditions at the forward variables of the option expiry

$$f_{n,n}(k \mid S_j) = \begin{cases} \max(S_n - k, 0) & \text{if it is a call option to buy} \\ \max(k - S_n, 0) & \text{if it is a put option to sell} \end{cases}$$

as previously by unfolding the recursion. The backward recurrence of the option premium is the random walk equivalent to the PDE $V_t = rV - (\mu + \frac{1}{2}\sigma^2)\,S\,V_S - \frac{1}{2}\sigma^2 S^2 V_{SS}$ of the statistically unbiased and accurate premium for an option on a proportional Brownian motion market asset S for an option portfolio that has not been debased. When it has been debased of return with slack, this is the discrete equivalent to the BSM PDE for the debased option premium. With or without first debasing the premium, this equation is the essential item to obtain the portfolio return components over each step of the random walk; the accurate premium for the correct return components and the debased premium for the return components fixed flat at the cash rate. Similarly in continuous time.

As an example, for a call option for the four step walk

walk and option parameters			
s0	110	u	1.12
k	87.69133	d	0.892857
δ	0.05	p	0.8
r	0.051271		

asset

0	1	2	3	4
				173.0871
			154.5421	
		137.984		137.984
	123.2		123.2	
110		110		110
	98.21429		98.21429	
		87.69133		87.69133
			78.29583	
				69.90699

option

0	1	2	3	4
				85.3958
			74.55277	
		64.82193		50.29267
	56.09808		42.51603	
48.30889		35.58371		22.30867
	29.53638		16.97653	
		12.91886		0
			0	
				0

G. The discrete total derivative of the option premium *f*

From the backwards recursion of the option premium expressed in terms of the option delta, the forward premium becomes

$f_u = (1+r)\,f_0 + qS_0(u-d)h_0$ if the underlying spot price appreciates to S_0u and
$f_d = (1+r)\,f_0 - pS_0(u-d)h_0$ if the underlying spot price depreciates to S_0d

for which the forward change in the option premium is

$\Delta f_0 = f_u - f_0 = rf_0 + qS_0(u-d)h_0$ if the underlying spot price appreciates to S_0u and
$\Delta f_0 = f_d - f_0 = rf_0 - pS_0(u-d)h_0$ if the underlying spot price depreciates to S_0d

The total change in the underlying asset is

$\Delta S_0 = S_u - S_0 = S_0u - S_0 = S_0(u-1)$ if the underlying spot price appreciates to S_0u and
$\Delta S_0 = S_d - S_0 = S_0d - S_0 = S_0(d-1)$ if the underlying spot price depreciates to S_0d

i.e.,

$$\Delta S_0 = \begin{cases} S_0(u-1) \text{ if the market asset moves up} \\ S_0(d-1) \text{ if the market asset moves down} \end{cases}$$

So writing the total change in the option premium if the spot price of the market asset moves up to S_0u as

$$\begin{aligned} \Delta f_0 &= rf_0 + qS_0(u-d)h_0 - S_0(u-1)h_0 + S_0(u-1)h_0 \\ &= rf_0 + S_0(qu-qd-u+1)h_0 + S_0(u-1)h_0 \\ &= rf_0 - (pu+qd-1)S_0h_0 + h_0\Delta S_0 \end{aligned}$$

and similarly if it moves down to $S_0 d$

$$\begin{aligned}\Delta f_0 &= rf_0 - pS_0(u-d)h_0 - S_0(d-1)h_0 + S_0(d-1)h_0 \\ &= rf_0 - S_0(pu - pd - d - 1)h_0 + S_0(d-1)h_0 \\ &= rf_0 - (pu + qd - 1)S_0 h_0 + h_0 \Delta S_0\end{aligned}$$

Hence, the total change/derivative in the option premium is

$$\Delta f_0 = rf_0 - (pu + qd - 1)S_0 h_0 + h_0 \Delta S_0 \qquad \text{where} \qquad \Delta S_0 = \begin{cases} S_0(u-1) \text{ with probability } p \\ S_0(d-1) \text{ with probability } q \end{cases}$$

which is **(1)** a cash rate increase from the decay in the discount factor on the premium *f*, **(2)** amortization of the premium at the market rate in proportion to delta and *S* for time decay in the deferral of the spot transaction [the derivative], plus **(3)** the forward binomial term for any option proper ($a > 0$), also in proportion to delta, for variability in the likelihood the option will be turned down [the option]. For the j^{th} step of the replicating portfolio, for arbitrary *j*, from the backwards recursion of the option premium, the forward premium becomes

$$\begin{aligned} f_{j,n}(k \mid s_0) &= (1+r) f_{j,n}(k \mid s_0) + q s_0 (u-d) h_{j,n}(k \mid s_0) && \text{if the underlying spot price appreciates to } S_j u \text{ and} \\ f_{j,n}(k \mid s_0) &= (1+r) f_{j,n}(k \mid s_0) - p s_0 (u-d) h_{j,n}(k \mid s_0) && \text{if the underlying spot price depreciates to } S_j d \end{aligned}$$

for which the forward change in the option premium is

$$\begin{aligned} \frac{\Delta f_{j,n}(k \mid s_0)}{\Delta j} &= f_{j+1,n}(k \mid s_0 u) - f_{j,n}(k \mid s_0) && \text{if the underlying spot price appreciates to } S_j u \text{ and} \\ &= rf_{j,n}(k \mid s_0) + q s_0 (u-d) h_{j,n}(k \mid s_0) \\ \frac{\Delta f_{j,n}(k \mid s_0)}{\Delta j} &= f_{j+1,n}(k \mid s_0 d) - f_{j,n}(k \mid s_0) && \text{if the underlying spot price depreciates to } S_j d \\ &= rf_{j,n}(k \mid s_0) - p s_0 (u-d) h_{j,n}(k \mid s_0) \end{aligned}$$

The total change in the underlying asset is

$$\frac{\Delta S_j}{\Delta j} = \begin{cases} S_j(u-1) \text{ if the market asset moves up} \\ S_j(d-1) \text{ if the market asset moves down} \end{cases}$$

so the total change in the option premium if the market asset moves up to $S_0 u$

$$\begin{aligned}\frac{\Delta f_{j,n}}{\Delta j} &= rf_{j,n} + qS_j(u-d)h_{j,n} - S_j(u-1)h_j + S_j(u-1)h_{j,n} \\ &= rf_{j,n} + S_j(qu - qd - u + 1)h_{j,n} + S_j(u-1)h_{j,n} \\ &= rf_{j,n} - (pu + qd - 1)S_j h_{j,n} + h_{j,n}\frac{\Delta S_j}{\Delta j}\end{aligned}$$

and similarly if it moves down to $S_0 d$

$$\begin{aligned}\frac{\Delta f_{j,n}}{\Delta j} &= rf_{j,n} - pS_j(u-d)h_{j,n} - S_j(d-1)h_{j,n} + S_j(d-1)h_{j,n} \\ &= rf_{j,n} - S_j(pu - pd - d - 1)h_{j,n} + S_j(d-1)h_{j,n} \\ &= rf_{j,n} - (pu + qd - 1)S_j h_{j,n} + h_{j,n}\frac{\Delta S_j}{\Delta j}\end{aligned}$$

Hence, the discrete total derivative in the option premium is

$$\frac{\Delta f_{j,n}}{\Delta j} = rf_{j,n} - (pu + qd - 1)S_j h_{j,n} + h_{j,n}\frac{\Delta S_j}{\Delta j} \qquad \text{where} \qquad \frac{\Delta S_j}{\Delta j} = \begin{cases} S_j(u-1) \text{ with probability } p \\ S_j(d-1) \text{ with probability } q \end{cases}$$

Net of the forward binomial term (in proportion to the delta), the per period return factor $pu + pd - 1$ and coefficient of delta and the spot variable is the return on the asset *S* that the option is written on and the asset upon

which the option depends, expires, and must be delivered if inthemoney so that it is not turned down. This may or may not be the cash rate, but it is in all sensible circumstances not the cash rate unless it has been purposefully demeaned of return as a result of debasing the option replicating portfolio with the introduction of slack which kills the return from $(pu+qd-1)S_j h_{j,n}$. The total derivative for the option premium should be compared with the total derivative of the replicating portfolio for which it can be seen that the only terms that differ are the split of the total change in the delta holding of the market asset $\Delta(S_j h_{j,n})$ into value changes and quantity changes; $h_{j,n}\Delta S_j$ for the total derivative of the option premium (which is killed by the offsetting asset) and $S_{j+1}\Delta h_{j,n}$ for the replicating portfolio as the outstanding delta rebalance due which must be made to complete the total overall derivative of the replicating dynamics for one complete step. Therefore, an *n* step transition to option maturity means *n* [high frequency] transactions in the underlying market asset to maintain delta for an unbroken replication, i.e., without introducing gamma and higher order terms of stochastic variation into the portfolio cashflows; terms of variation which cannot be neutralized.

Note that delta is not sufficient for any asset that differs in any way from that with a binomial basis function. Considering the next most simple discrete valued basis function of a walk, that of a trinomial walk; the total derivative of an option on a trinomial walk is

$$\frac{\Delta f_{j,n}}{\Delta j} = rf_{j,n} - (pu+qm+rd-1)S_j h_{j,n} - (pu^2+qm^2+rd^2-1)S_j^2\Gamma_{j,n} + h_{j,n}\frac{\Delta S_j}{\Delta j} + \Gamma_{j,n}\frac{\Delta S_j^2}{\Delta j}$$

where $\Gamma_{j,n} = \dfrac{\delta^2 f_{j+1,n}}{\delta S_{j+1}^{\ 2}}$ is a gamma hedge ratio and the total derivative therefore depends on ΔS_j^2.

It is not possible to kill a stochastic change in the square of the market asset ΔS_j^2 by taking an offsetting position in S^2 since investing in S^2 is impossible so it is impossible to neutralize gamma and all higher order effects of stochastic variation. This applies not only to trinomial walks, but also to binomial walks with missed delta rebalances [as proved, and here again] and hence voided replicating arguments for anything short of *n* rebalancing transactions for an *n* step maturity whereby the missed delta updates accumulate into gamma effects, i.e., with any single missed market fluctuation. Given the total derivative for trinomial walks, and dependence upon ΔS_j^2, the same issues are then accentuated with every additional node in the basis function so the inability to neutralize variation from a replication, net of the signal of trades maintaining delta, includes tetranomial and all other multinomial walks, i.e., every single other form of random walk apart from that with a binomial basis function. This inability to neutralize a portfolio from random variation applies equally to debasing a replicating portfolio whereby subsidizing the mispricing in the debased premium for zero NPV of cashflow is entirely voided and outright wrong for any variable that actually trades in the financial markets, including any form of continuous variable. Nice maths though.

H. The total derivative, backward recurrence, and return components of the option portfolio Π

The option replicating portfolio is $\Pi_0 = S_0 h_0 - f_0$ where $S_0 h_0$ replicates f_0. The total change in the portfolio $\Delta\Pi_0$ over the first step is then

$$\begin{aligned}\Delta\Pi_0 &= \Delta(h_0 S_0) - \Delta f_0 \\ &= S_1\Delta h_0 + h_0\Delta S_0 - \Delta f_0\end{aligned}$$

and substituting the total change of the option premium

$$\Delta f_0 = rf_0 - (pu+qd-1)S_0 h_0 + h_0\Delta S_0 \qquad \text{where} \qquad \Delta S_0 = \begin{cases} S_0(u-1) \text{ with probability } p \\ S_0(d-1) \text{ with probability } q \end{cases}$$

into the total portfolio change

$$\begin{aligned}\Delta\Pi_0 &= S_1\Delta h_0 + h_0\Delta S_0 - \left(rf_0 - (pu+qd-1)S_0 h_0 + h_0\Delta S_0\right) \\ &= S_1\Delta h_0 + (pu+qd-1)S_0 h_0 - rf_0\end{aligned}$$

This shows that, net of the outstanding delta rebalance $S_1\Delta h_0$, the portfolio that replicates f_0 with $S_0 h_0$ returns the market return $pu+qd-1$ on the market asset and incurs the cash rate *r* on the short option. This,

$\Delta\Pi_0 \neq r\Pi_0$, is not a fixed portfolio return flat at the cash rate, i.e., not $\Delta\Pi_0 = r\Pi_0$. Moreover, for a call option $\Delta\Pi_0 > r\Pi_0$ and holding $\Delta\Pi_0 = r\Pi_0$ does nothing other than debase the portfolio and the resulting backed out option.

For arbitrary step *j*, the option replicating portfolio that has been rebalanced at time *j* is $\Pi_{j,n} = h_{j,n}S_j - f_{j,n}$. The total change in the portfolio $\Delta\Pi_{j,n} = \Delta(h_{j,n}S_j) - \Delta f_{j,n}$ with respect to the backward variables is then

$$\begin{aligned}\frac{\Delta\Pi_{j,n}}{\Delta j} &= \frac{\Delta(h_{j,n}S_j)}{\Delta j} - \frac{\Delta f_{j,n}}{\Delta j} \\ &= S_{j+1}\frac{\Delta h_{j,n}}{\Delta j} + h_{j,n}\frac{\Delta S_j}{\Delta j} - \frac{\Delta f_{j,n}}{\Delta j}\end{aligned}$$

by the product rule of a discrete difference. The first term is a quantity change while the last two terms are value changes. Substituting in the expression for the total change in the option premium

$$\frac{\Delta f_{j,n}}{\Delta j} = rf_{j,n} - (pu + qd - 1)S_j h_{j,n} + h_{j,n}\frac{\Delta S_{j,n}}{\Delta j} \qquad \text{where} \qquad \Delta S_j = \begin{cases} S_j(u-1) \text{ with probability } p \\ S_j(d-1) \text{ with probability } q \end{cases}$$

for

$$\begin{aligned}\frac{\Delta\Pi_{j,n}}{\Delta j} &= h_{j,n}\frac{\Delta S_j}{\Delta j} - \left(rf_{j,n} - (pu + qd - 1)S_j h_{j,n} + h_{j,n}\frac{\Delta S_j}{\Delta j}\right) + S_{j+1}\frac{\Delta h_{j,n}}{\Delta j} \\ &= (pu + qd - 1)S_j h_{j,n} - rf_{j,n} + S_{j+1}\frac{\Delta h_{j,n}}{\Delta j}\end{aligned}$$

after canceling the replicating terms in the offsetting deltas. Net of the outstanding j^{th} delta update, $S_{j+1}\Delta h_{j,n}/\Delta j$, the total change in the portfolio is

$$\frac{\Delta\Pi_{j,n}}{\Delta j} = (pu + qd - 1)S_j h_{j,n} - rf_{j,n}$$

Hence, the portfolio return is composed of a change in **(1)** the value of the offsetting market asset *hS*, for a fixed delta, less **(2)** the change in the option premium *f* and **(3)** the cancelation of the replicating effects *h*Δ*S* in *hS* against that in *f* with the cancelation of the matched and offsetting deltas. It is seen clearly that the portfolio earns the market return $pu + qd - 1$ on the market asset *S* and incurs the cash rate *r* on the option.

From the example, the portfolio returns are the right hand lattice which are neither fixed nor flat at cash

asset

0	1	2	3	4
				173.0871
			154.5421	
		137.984		137.984
	123.2		123.2	
110		110		110
	98.21429		98.21429	
		87.69133		87.69133
			78.29583	
				69.90699

option

0	1	2	3	4
				85.3958
			74.55277	
		64.82193		50.29267
	56.09808		42.51603	
48.30889		35.58371		22.30867
	29.53638		16.97653	
		12.91886		0
			0	
				0

portfolio returns

0	1	2	3	4
				0.096288
			0.094387	
		0.09257		0.086849
	0.090973		0.08536	
		0.084369		0.079441
			0.079441	
				0

Similarly to the results in continuous time, a debased option portfolio holds the portfolio return flat at the cash rate and then backs out the differential equation (BSM equation) pertaining to the market asset demeaned of return as a result of simply forcing the solution of the asset return $pu + qd - 1$ equated to the cash rate (by adjusting a parameter for the slack) because of the additional and artificial constraint $\Delta\Pi_{j,n} = r\Pi_{j,n}$. This constraint simply debases the option portfolio and the resulting backed out option premium, and turns gold into lead, i.e., turns options into redundant securities and voids all cashflows of market return.

The proof is complete. Quod Erat Demonstrandum, as was to be proved.

A Consistent Framework for Derivatives

This thesis is arguing against the debasement derivatives, including the debasement of futures which, as it stands, trade for zero premium whatever the growth dynamics of the asset of the deferred spot transaction. This denies the economic value of a future, as a financial derivative and extension to the spot markets. It is asserted that premium and return is received for the participation in (and claim to) the economic activity of the asset for which the derivative is deferring, for example, going long in the asset to take up the interest for a deferred transaction of the spot markets by making a market in the future.

As a consistent framework for derivatives and consistency between futures and options; futures are treated similarly as options but without the max functions (options are futures but with the right to turn down the future transaction). The futures markets should have premiums and a negotiable futures price, a value which is distinct from the premium to enter into the position. The premium for a future is then the put-call parity of the equivalent option, for parity between the option price and futures price to deliver at maturity. Both are based upon deferring a spot transaction, the sole difference between the two is that one transaction at the maturity date is optional.

- a future … *k* is the **future price** and *f* is the **premium** for (the right and) the obligation to trade the asset at the future price *k* at the maturity date.
- an option … *k* is the **optional price** and *f* is the **premium** for the option (the right but not the obligation) to trade the asset for the optional price *k* at the maturity date.

It is suggested that the party taking the long side of the future pay the premium and that taking the short side of the future receive the premium (for a market asset with positive market return); the premium is then **(1)** that of put-call parity for the equivalent statistically accurate option, **(2)** the price of deferring the transaction for the underlying economic asset until the futures date, and **(3)** the estimate of the future's basis at maturity within a bid-ask spread for the interaction between supply and demand. The futures price is then variable (at the discretion of the parties involved and is a parameter of the agreement) for the futures agreement and it is the premium that is exchanged that is the value of the agreement rather than, as per current practice of the derivative exchanges, a fixed futures price so that the premium for the transaction is zero. This is efficient and more flexible, and consistent with a derivatives market that is an extension to the spot markets and consistent with options. Importantly, this can give **explicit recognition to cycles and seasonality in commodity prices**.

To demonstrate why a zero premium at the spot price is not a sensible default price for a future, think about the following situation; Is the spot price a sensible price for a brand new car if it is going to be delivered in 3 months' time whilst someone is driving it around for those three months? This is equivalent to a futures agreement on a commodity that is moving into a seasonal trough in the price variable, and in this case, denies the consumption value of the asset. This is analogous to the economic value of a financial variable; the depreciation from consumption of the asset [the car] is analogous to negative market drift [in a financial variable]. If a premium is exchanged with regard to the future, then the premium for the deferral of the transaction [the future] is then the value that is passed up for getting ownership of the car [the security] in three months' time rather than on the spot. The premium is the revenue for the service provided by the derivative in deferring the transaction.

Premium calculations for **both futures and options** require an assessment of **future return**, while **options** also **require probability**, since it depends on the probability that the option is inthemoney at expiry (or equivalently, the probability that the transaction will be turned down because of an unfavourable price in the asset *S*). This probability must be accurate if it is to be sensible. For futures in the context of statistically accurate premiums, the premium is smaller for a future because it has both upside and downside, i.e., without the asymmetry of options. A future is not as valuable as an option and therefore has a smaller premium for the long side (for $pu+qd > 1+r$ otherwise the premium is negative so the long side pays a premium). There is less margin for error if the markets were to try it out. Premiums would be set by reference to price data from the derivative markets, supply and demand, and by reference to probability theory and historical data from measured variables [and, importantly, for seasonality and harmonics with higher order diffusion variables].

If a premium is exchanged between the counterparties to the future, which takes up the value of the interest in the economic asset, then the dynamics of the partial difference equation of the premium is **a margin on services method of allocating the earned premium** (revenue to one side and an expense to the other) in proportion to the services provided by the derivative throughout the period of time between the futures agreement and exchange of premium and the future transaction. Further mathematical details to this are provided in DERIVATIVE AMORTIZATION SCHEDULES but the one important equation is

$$\frac{\delta f_{j,n}}{\delta j} = rf_{j,n} - (pu + qd - 1)S_j h_{j,n}$$

which allocates the premium as it is earned via the delta, i.e., the differential market return net of cash and amortization of the deferred transaction of the derivative. This method, with an adjustment to the PDE, can also take into account harmonics for market cycles and seasonality. Delta rebalances need not be at every single market fluctuation (*n* in total for the binomial variable), since gamma and higher order effects are zero due to a deterministic delta which is free from binomial CMFs and consequently do not accumulate into gamma effects from missed rebalances. For example, the method is still applicable if every second or every second and third delta update is missed as it is just made up for with the updates that are made. This is not possible with options proper for which there is uncertainty as to whether the future transaction will take place and hence randomly variable deltas dependent upon ΔS and accumulating gamma effects for every missed binomial rebalance.

A **pseudo option**[8] is a future with a non-sensible futures price *k* whereby there is a large probability for a near certain payoff (100% certain for a proper pseudo option). For example, with $k = 0$, a pseudo-call is the right but not the obligation to get the asset for nothing at the expiry date. Of course it will be exercised so it has redundant optionality, i.e., zero probability that it will be turned down and the outcome does not depend on probability. **A pseudo option** is a form of **fringe option**[9]. The other form of fringe option are **lottery options**, which are bets with small probabilities for large payoffs. **Mainstream options** are options with sensible exercise prices at or near the money, i.e., the exercise price *k* is at or near the spot price S_0. Mainstream options are always **options proper** and the right to turn down an unfavorable exchange always has value.

A future is a **put-call parity combination of options proper**. A long future is a long call with a short put whilst a short future is a long put with a short call. A futures price *k* of a sensible future, therefore, is also around/near the spot price S_0, and the premium is smaller than that of the option from the put-call parity relationship which expresses the premium for the future. The difference between the future's premium (and no premium) is then the economic value of the asset for which the short side of the future is taking an interest in (for the long side) until the date of the deferred transaction at the futures date i.e., shorting the future and then long in the asset to take up the pecuniary interest in the economic asset until the deferred transaction of the future. This is the value of the futures agreement and derivative as [derived from] a deferred transaction of the spot markets.

If you are looking to cash in a premium, looking for pseudo options is a good place to start. This means options that are inthemoney and will stay inthemoney, which are low exercise price calls and high exercise price puts for inthemoney options on low volatility market assets. These [pseudo options] are inefficiencies. If you do return a profit, then this should be because you have taken a correct interest in the market [and contributed to making it more efficient] rather than notionally having exploited the market for an arbitrage. Attempting to bar return and profit is doing nothing other than attempting to take away all incentives to write and trade derivatives, and discouraging any notion of being accurate and efficient.

Options proper have just been defined as options that are atthemoney or very nearthemoney. Then, by this definition, options proper always have high gamma; gamma is then a defining characteristic of an option proper. Only pseudo options are zero gamma options. Zero gamma is a defining characteristic of futures and pseudo options. The gammas of the put-call combination of a future cancel which is why futures have zero gamma.

options proper $\Rightarrow$ atthemoney or very nearthemoney
$\Rightarrow$ stochastic delta
$\Rightarrow$ variable and rapidly changing gamma [and risky by definition]

This means that

- large gamma implies large risk in delta updates
- large gamma implies large accumulations of gamma with missed updates
- zero gamma implies a pseudo option
- options proper always have gamma which means that options proper always have risk [stochastic variation in *f* whether it is replicated and offset or not, no exceptions]

■

The difference change in the portfolio replicating the accurate premium, net of changes in the delta, has been proved to be determined by the interaction between the growth in the underlying market asset and the cash rate and consequently not flat from step to step. Inclusive of the delta update $S_{j+1}\Delta h_{j,n}$ at the node, the total portfolio change is

$$\Delta\Pi_j = (pu + qd - 1)h_j S_j - rf_j + S_{j+1}\Delta h_j$$

[8] as in a pseudo pin from chess.
[9] as in on the fringe of mainstream.

For the hypothetical binomial market asset where ΔS_j is sufficient to kill the option, the return components of the binomial replicating portfolio that is not debased and is instead **free to earn market return** $\Delta\Pi_j = (pu + qd - 1)h_j S_j - rf_j + S_{j+1}\Delta h_j$ should be compared with the debased option portfolio containing the debased premium that is statistically unbiased and accurate only for the demeaned asset

$$\begin{aligned}\Delta\Pi_j &= rh_j S_j - rf_j + S_{j+1}\Delta h_j \\ &= r\Pi_j + S_{j+1}\Delta h_j\end{aligned}$$

which is obtained by demeaning the market asset of its market return by varying either the probability of an up-step or the return growth factor so that either **(1)** the debased asset earns an average effective rate of r rather than $pu + pd - 1$ (i.e., earns only the cash rate but has full market risk) or **(2)** the mispricing in the debased option premium sinks the market return on the market asset keeping the portfolio return flat and debased at r. Note that buying a put is synonymous in derivatives to shorting an asset. Puts should be less commonplace not because they are overpriced (after being debased) but because they are a position on the wrong side of a market with economic growth, **entirely analogous to the spot markets whereby short trades (and short traders) comprise only a minority of market trades since it is taking a position contrary to the economic growth pattern of markets**.

The results deduced for components of portfolio return are now used to derive an expression for the cashflows of the option replicating portfolio, and then contrasted to the cashflows of the portfolio debased by constraining the portfolio return fixed and flat at the cash rate. In the context of taking an offsetting position in an offsetting asset to kill the option, these results show explicitly that the discounted bias in the debased premium is then a subsidization arising from the opportunity cost of earning zero excess return on trading the underlying market asset S to kill the option and subsidize the option mispricing. In other words, the net expected gains from killing the option by trading in and replicating the option with an offsetting asset that grows by more than a bank account is used to reduce the premium of a call option and visa-versa for a put option, the more expensive put option pays for the losses on the short position in the asset. If the effective return of the market asset is less than the cash rate, then the opposite holds in that the direction of the subsidy between trading returns and premium is reversed. Given that **there is nothing to show for flatlining the option**, and risk-neutral valuation states that all options must be flatlined, a risk-neutral market is better shut down since it is **more efficient** to shut a risk-neutral market down, zeroing all cashflows and achieve the same outcome ***(if it worked, which it doesn't)***. Note that **without replicating and killing the option, there are no means by which to subsidize the mispricing from debasing an option portfolio, and the resulting debased premiums are outright wrong.**

NPV of the Cashflows of the Replicating Portfolio for the Statistically Unbiased Option and Market Asset S

For the replicating portfolio with the accurate and statistically unbiased premium, it is now shown that the net-present-value (NPV) of cashflows is a profit for replicating and nullifying a written call and a loss for replicating and nullifying a written put, and further, that this profit or loss is equal to the excess returns, over the cash rate, earned on the period-to-period asset holdings in proportion to the replicating delta. In particular, a **(i)** profit earned on trading with a net long position in the market asset S to flatline the call option or **(ii)** loss incurred on trading with a net short position in the market asset S to flatline the put option. The position in the asset and return on cashflow amounts to taking up the interest in the economic asset for the deferred spot transaction of the derivative. This result regarding return on assets also applies to replicating and nullifying a purchase, rather than a sale, of an option.

The statistically accurate option premium,

$$f_{j,n} = v^{n-j}E\left(\max(S_n - k, 0) \mid S_j\right) \qquad \text{for a call option or}$$

$$f_{j,n} = v^{n-j}E\left(\max(k - S_n, 0) \mid S_j\right) \qquad \text{for a put option}$$

is replicated and offset by trading the underlying market asset S to deliver in proportion to delta. The relation of the differential changes between the binomial nodes of the portfolio is the total time derivative of the replicating portfolio. This is the return components plus the delta update

$$\frac{\Delta\Pi_j}{\Delta j} = (pu + qd - 1)h_j S_j - rf_j + S_{j+1}\frac{\Delta h_j}{\Delta j}$$

i.e.,

$$\Delta(h_j S_j - f_j) = (pu + qd - 1)h_j S_j - rf_j + S_{j+1}\Delta h_j$$

This is a first order linear difference equation in the value of the replicating portfolio and the discrete backward time variable for the discrete value change in the portfolio, which moves forward with an increase in the backward time variable as expiry at the forward time variable n nears. After deducting $rh_{j,n}S_j$ from both sides in order to put the expression in terms of present values and to bypass the requirement for summation by parts, this can be written as

$$\Delta(h_j S_j - f_j) - r(h_j S_j - f_j) = (pu + qd - 1 - r)h_j S_j + S_{j+1}\Delta h_j$$

i.e.,

$$\Delta\Pi_j - r\Pi_j = (pu + qd - 1 - r)h_j S_j + S_{j+1}\Delta h_j$$

The backward variable j is the independent variable. The integrating factor for the terms of the portfolio $\Delta\Pi_j - r\Pi_j$ is then the discount factor $v^{j+1} = (1+r)^{-(j+1)}$ which has the effect of discounting both sides at the cash rate r to the time the portfolio is established at option inception, i.e., puts all terms in present values

$$v^{j+1}\Delta\Pi_j - r\,v^{j+1}\Pi_j = (pu + qd - 1 - r)v^{j+1} h_j S_j + v^{j+1} S_{j+1}\Delta h_j$$

and the portfolio changes can be factorized in Δ

$$\Delta\left(v^j \Pi_j\right) = (puv + qdv - 1)v^j h_j S_j + v^{j+1} S_{j+1}\Delta h_j$$

Summing from the establishment of the portfolio to option expiry over the backward time variable j

$$\sum_{j=0}^{n-1} \Delta\left(v^j \Pi_j\right) = (puv + qdv - 1)\sum_{j=0}^{n-1} v^j h_j S_j + \sum_{j=0}^{n-1} v^{j+1} S_{j+1}\Delta h_j$$

Noting that the factorized portfolio changes are telescopic with Δ (a discrete integrated derivative) so that

$$\begin{aligned}\sum_{j=0}^{n-1} \Delta\left(v^j \Pi_j\right) &= \left(v\Pi_1 - \Pi_0\right) + \left(v^2\Pi_2 - v\Pi_1\right) + \ldots \\ &\quad \ldots + \left(v^{n-1}\Pi_{n-1} - v^{n-2}\Pi_{n-2}\right) + \left(v^n \Pi_n - v^{n-1}\Pi_{n-1}\right) \\ &= v^n \Pi_n - \Pi_0\end{aligned}$$

and the summed components of return can be written as

$$v^n \Pi_n - \sum_{j=0}^{n-1} v^{j+1} S_{j+1}\Delta h_j - \Pi_0 = (puv + qdv - 1)\sum_{j=0}^{n-1} v^j h_j S_j$$

which is the equation of the NPV of cashflows. This can alternatively be expressed in terms of the components of the portfolio

$$v^n (S_n h_{n,n} - f_{n,n}) - \sum_{j=0}^{n-1} v^{j+1} S_{j+1}\Delta h_{j,n} - (S_0 h_{0,n} - f_{0,n}) = (puv + qdv - 1)\sum_{j=0}^{n-1} v^j S_j h_{j,n}$$

where $f_{0,n} = v^n E\left(\max(S_n - k, 0) \mid S_0\right)$ and $f_{n,n} = \max(S_n - k, 0)$ for a call and $f_{0,n} = v^n E\left(\max(k - S_n, 0) \mid S_0\right)$ and $f_{n,n} = \max(k - S_n, 0)$ for a put. This shows that for a call option ($h_j \geq 0$ and assuming that $pu + pd > 1 + r$) the total net cashflows, arising from **(i)** establishing, **(ii)** rebalancing, and **(iii)** closing out the replicating portfolio, results in a profit from these hedging activities (in return for the option payoff profile) and that this profit is equal to the excess return, over the cash rate, earned on the market asset traded within the portfolio over the lifetime of the option

closing out proceeds $-$ *initial cost* $-$ *rebalancing costs*

$=$

excess return on market assets

for which the option payoff profile is traded for market returns on the traded market asset S that offsets and nullifies the option. For a put option, the delta has the opposite sign ($h_j \leq 0$) and the total NPV of cashflow results in a loss for $pu + pd > 1 + r$. That is, whenever the mean effective return on the underlying asset is greater than the cash rate $pu + pd - 1 > r$, the hedging portfolio of the unbiased and statistically accurate premium is profitable for a call option and loss making for a put option (since the asset is traded long for the call and short for the put), the extent of the profit or loss being the extent to which the earnings on the portfolio market asset S ex-

ceeds the cash rate. This is **consistent with profiting from taking a correct position in the economic growth pattern of the financial markets** (which is sourced from growth in GDP) and taking up the pecuniary interest in and claim to the economic activity of the economic asset for the deferred transaction of the derivative. The equation of the NPV of cashflows has the same form as that with the demeaned asset except for asset returns, i.e., all paths lead to positive (negative) asset returns from trading long (short) the excess returning market asset. The coefficient on the present value of the asset holdings $vpu + vpd - 1 = v\left(pu + pd - 1 - r\right)$ is the excess return earned, over the cash rate, on the assets over any single period and step in *S*. The equation of the NPV of cashflow becomes that of the simpler case of the debased portfolio whereby the portfolio return is fixed flat at the cash rate, and the demeaned asset is traded when the asset return $pu + pd - 1$ is demeaned to the cash rate r so that the coefficient of the term on the right zeros

$$p'u + q'd - 1 - r = 1 + r - 1 - r = 0$$

and the rate of return on the underlying asset, $pu + pd - 1$, is replaced by the cash rate, r. The term on the right of the NPV of cashflows, which is returns earned on the market asset, vanishes (sunk by either a demeaned asset or a mispriced option) leaving the outcome under the debased portfolio containing the option priced for the equivalent demeaned asset void of market return

$$(p'u + q'd - 1 - r)\sum_{j=0}^{n-1} v^{j+1} h_j S_j = (1 + r - 1 - r)\sum_{j=0}^{n-1} v^{j+1} h_j S_j = 0 \times \sum_{j=0}^{n-1} v^{j+1} h_j S_j = 0$$

That is, the profit on the investment earnings from trading the market asset *S* to replicate, offset, and null the option [kill the payoff profile] is used to reduce the call option premium or, in the alternative circumstance, increase the put option premium to pay for the investment losses on short trading in *S*, i.e., to subsidize an overpriced or underpriced option. The option is therefore killed (by the offsetting asset) and the returns earned (with the asset) on killing the option are used to subsidize it (!) by flatlining the option and turning it into a redundant security. This is not consistent with a sensible market rationale nor consistent with a prosperous market for options [or other financial derivatives] and is one of several items that comprise the enigma of risk-neutral valuation and the illusion of a biased and debased set of premium formulae being appropriate "fair" prices, which they clearly are not.

As will be shown in the following section, the term of the present value of the market return earned on the replicating market asset, which nullifies the option within the hedging portfolio [by killing its delta], could hypothetically be moved to the other side of the equation and used to reduce the initial option premium f_0. This is letting the excess returns from trading the underlying asset accrue to the purchaser of the option rather than the portfolio. In this case, something very close to the debased option price is obtained. The remaining difference is the differences in the size and changes of the deltas. Because of the direction of the mispricing in the debased premiums, it is wise to buy call options and shun put options that are traded at debased prices; an option house that writes these options faces certain ruin unless it compensates in some way such as skewing volatility[10] (or skewing the bid-ask spread) with ignorance of the statistically accurate price.

B. THE DEBASED REPLICATING PORTFOLIO OF RISK-NEUTRAL VALUATION AND THE DEBASED PREMIUM

Introduction

This section discusses the debasement of an option replicating portfolio, the resulting distribution of the demeaned asset *Y*, and the implications of pricing an option for the demeaned asset *Y* rather than the market asset to deliver *S*. This price, which is accurate for *Y* rather than *S*, is the debased price which is statistically biased for the market asset to deliver. This statistical bias and inaccuracy is introduced into the option premium whenever and however an option portfolio is debased of market return. The greater the market return, the greater the effects of debasing the portfolio and premium. The formulae for the debased premium and delta, for the options priced for the demeaned asset, are stated as modifications of the premium of the statistically unbiased and accurate option, and the backwards recurrence equation of the debased option price is deduced by debasing the return from the results of that of the market asset and accurate premium. This is then used to show that the between node portfolio returns are fixed at the cash rate, and flat from period to period, as a result of debasing the

[10] which is an inefficient method of trying to bring the debased price back to the accurate price by reference to the market; i.e., pricing the slack into volatility which has been debased from return. The better method is by reference to the markets assessment of market return via [undebased] put-call parity and adjusting the spot price accordingly, i.e., **matching market put-call parity (which is the premium for the future)**.

portfolio Π with the constraint placed on Π, and the resulting demeaned asset for which the debased option is then priced earning the same rate as that for which the short option incurs net of the replicating delta and the delta rebalances. This is then both the implication and the consequence of holding the return fixed and flat with $\Pi_0 = v\Pi_u = v\Pi_d$ which surreptitiously demeans the market asset *S* creating the demeaned asset *Y* and a debased option premium [turning gold into lead]. The notional asset *Y* is referred to as either **(1)** the demeaned asset or **(2)** the debased asset or **(3)** the oxymoron risky bank account or **(4)** zero returning asset or **(5)** a redundant asset or **(6)** the return-castrated asset or **(7)** a defunct asset. The distribution of the oxymoron risky bank account *Y* is referred to by the literature as **the risk-neutral measure which is a complete misnomer.** The option accurate for the demeaned asset is a debased premium, and the difference between the debased premium and that which is statistically accurate for the market asset to deliver is the statistical bias and inaccuracy in the debased premium. Unlike the continuous time arguments, the variance characteristics also change depending on how the market asset is demeaned to debase the portfolio and premium. Debasing by modifying the probability *p* to change the return profile is inconsistent with accepted financial wisdom that there is a positive relationship between risk and return. The symbol *S* is still the actual asset upon which the option expires and its distribution remains unchanged since it is exogenous, although the debased premium is priced as though it was written on *Y* and hence void of market return. The difference equation of the return components of the portfolio containing the option priced for the demeaned asset *Y* (the debased option) are then summed into the expression for the discounted portfolio cashflows occurring over the life of the portfolio and the option. This is compared to that for the portfolio that replicates the option premium which is accurate for the market asset *S* by trading the market asset *S*. This is discussed in the context of **(1)** a replication of the premium and option accurate priced for *Y* by trading and rebalancing with the notional demeaned asset *Y* and then **(2)** a replication of the premium and option priced for *Y* by trading and rebalancing with the market asset *S*. From the results of integrating the difference equation of portfolio returns, it is proved that when the excess returning market asset is traded to replicate and offset of the debased option, all of the excess returns earned on the period-to-period holdings of the market asset do not accrue to the portfolio but instead go entirely to subsidizing the mispricing in the debased option premium. The return on trading the offsetting market asset is used entirely to subsidize the mispriced option whereby the mispriced option premium, priced for *Y*, sinks the trading returns earned on *S*. The net effect? sunk trading returns and a sunk option. The replicating and offsetting asset kills all of the return as well as the variation and turns the option into a redundant security void of a payoff profile. The expected values of the portfolio establishment, rebalancing, and closing out transactions as the items of the portfolio cashflows are obtained which further explains the inaccuracy from the statistical bias in the debased option price in the context of replicating the delta of an option with the offsetting asset which **swaps** trading returns (and arduous labour and blood, sweat, and tears of intensive trading) for a flatlined option payoff profile. The outcome? no gain [with certainty] for a lot of pain [hard work]. If the option is not replicated and killed by an offsetting asset, for example if someone wants the option payoff profile and does not replicate to kill it, then the only sensible and accurate value for the option is the statistically unbiased premium for the market asset to deliver *S* since there is then no means by which to subsidize the mispricing in the debased premium. This is also the sensible price even if one is considering replicating in which case, since if one can accurately judge the correct side of the market, then there is no reason why one should not be allowed to earn returns and profit by taking a correct economic interest in the market asset and earn a return by trading *S* in return for making a market in the derivative. It is also the correct price because it is accurate for the option and also because the replicating arguments are contrived in that there must be a rebalance at every single node (every step of the walk and fluctuation in the market which is physically and financially impossible) otherwise the portfolio does not remain delta-neutral and the replication is broken so that the replicating arguments are void from accumulated gamma. The outcome of the replication in all circumstances is then a skewed payoff profile of an indeterminate form, from the accumulation of unhedgeable gamma terms, that is something between **(1)** that of the option and **(2)** a covered call in the case of a call option and a protected put in the case of a put option. In actual markets, due to an infinite number of missed rebalances and a void replication, and the fact that neutralizing instantaneous delta is not sufficient to neutralize market risk, there is so much sampling variation [and therefore will always be so far off the mark from zero NPV] that the subsidization arguments for sourcing mispricings in the debased premia to keep portfolio returns flat and void of return are entirely defunct. It is impossible to neutralize every single market fluctuation because of frequency, entirely apart from the fact that neutralizing instantaneous delta is not sufficient to neutralize any single market fluctuation. Even if it worked which it doesn't, the outcome of a replication of a debased premium is then nothing but a redundant flatlined option, which risk-neutral says must happen for all options. It is therefore more efficient for a risk-neutral option market to be shut down, then all option payoff profiles are killed and all cashflows zeroed without having to do a hell of a lot of trading for absolutely no benefit [but failing every time]. The sensible alternative? accurate premiums and derivative markets that are explicitly recognized as an extension to the spot markets by fully recognizing return [in the context of probability theory]. This is the purpose of BOOK 2 of this PhD thesis.

Debasement of the Option Portfolio; the Demeaned Asset Y, and the Option Priced for the Demeaned Asset Y

For the two-valued binomial variable and the portfolio where the quantity of the market asset purchased and held against the option to offset and nullify the option [to kill its delta] is maintained in proportion to delta, the net portfolio change is then composed of one deterministic term and one stochastic term. The former is independent of whether the underlying market asset moves upwards or downwards ($\Pi_u = \Pi_d$ for two-valued binomial risk, net of the update) and the latter term is the [risky] delta update due which is required to maintain the randomly variable and changing momentary delta to keep it matched. With this offsetting position at the start of the binomial step, the portfolio is delta neutral and, equivalently, duration matched for market dependent cashflow. It was proved (by several methods) that for an accurate premium for the market asset *S*, the deterministic term is composed of the market return on the delta position in the market asset *S* and the cash rate on the short option *f*. Contrary to this, and in additional to the delta neutral condition of $\Pi_u = \Pi_d$ for the killed option delta, the return-neutral methodology debases the portfolio by placing an additional restriction that the portfolio return is held fixed flat at the cash rate $(1 + r)\,\Pi_0 = \Pi_u = \Pi_d$ after neglecting the economic value of the derivative and not giving explicit recognition to the update that must be made if the portfolio is to remain delta-neutral and replicate the option. This extra algebraic constraint $v\,\Pi_1 = \Pi_0$ that the return-neutral methodology imposes is at the expense of introducing inaccuracy and statistical bias, and is artificial and misguided, and debases the option portfolio turning gold [derivatives] into lead [redundant securities]. As it was proved that with the accurate premium for the market asset *S*, the position in the market asset *S* earns the market return of *S* and incurs the cash rate on the option premium, the consequence of this additional restriction is then a mispriced and wrong premium that is correct only for oxymoron risky bank accounts (debased market assets void of return μ). Since the additional constraint does not correspond to the correct underlying asset and the premium of that asset, it **forces a mispricing in the option premium** which sinks the return on the traded asset *S*. The replication then sinks the option. The debased premium is only statistically unbiased and accurate for the demeaned asset *Y*. Risk-neutral valuation calls the surreptitious transfer that results by debasing the portfolio $(1 + r)\,\Pi_0 = \Pi_u = \Pi_d$ a "change of measure" which simply removes (castrates) the excess return **and debases the premium (turning it into a mispriced security)** by setting $pu + qd$ to $1 + r$ and backing out *p*, i.e., debasing the asset and premium by varying the parameter *p* to take up the slack Δ between return and no return. For the binomial asset, this also increases the variance so that the premium is priced for 0% market return and more than 100% of the market risk. T**he replication which subsidizes the mispricing then nullifies the option and turns the option into a redundant security.**

The distribution of the demeaned asset *Y* is referred to by the risk-neutral literature as the risk-neutral probability measure, which is a misnomer, and is the distribution of the underlying asset where all return above the cash rate is removed with a slack variable Δ. An option portfolio need not be debased of return for it to be delta-neutral, so the distribution remains that of the market asset *S* whenever the portfolio and premium have not been debased. Noting that the position in the delta-neutral portfolio is not risk free (since there is always risk outstanding), the asset is demeaned and debased of return by constraining the portfolio return flat at the cash rate by assuming that **(1)** there is no risk after pretending that delta is constant [while not so much of an issue for two valued risk, it is so for any other form of risk including three valued risk of trinomial walks, four valued risk of tetranomial walks, five valued risk of pentanomial walks, …] and **(2)** assuming that there is no economic value to derivatives. This constraint $\Pi_0 = v\Pi_u = v\Pi_d$ surreptitiously debases the asset and the consequent premium *f* for the option to trade *S*, and this premium is statistically biased and inaccurate to the degree that it differs from the statistically accurate and unbiased price for the option on the market asset to deliver, and the degree to which it has been debased from the benefits and economic prosperity of the financial markets. **Risk-neutral valuation is a term that is therefore synonymous with the debasement of derivatives[11].**

Taking the value of the portfolio between the first and the second node (or any two successive nodes) of the lattice. With the artificial condition that the portfolio return is held flat at the cash rate, at the end of the step just before the mandatory at-node delta update, then by this restriction and artificial constraint the portfolio is de-

[11] **similarly to coinage in the early days of money and finance;** … a debased coin is not an accurate measure of value so that when a debased coin [option premium] is exchanged for a good or service [a deferred spot transaction], the exchange is whence a biased and skewed exchange of value… a single such transaction may not have noticeable ramifications, but a large number of such transactions are bound to have very serious economic ramifications which are not discernable to each individual but to whom are each affected as a whole [the economy] with aggregate effects and consequences such as inflation in purchasing power and the cost of financing.

based to a single cash rate accumulation factor larger than that at the start of the step (which keeps up only with cash[12], a non-performing asset)

$$\Pi_0 = v\Pi_u = v\Pi_d$$

This is the constraint that debases the option portfolio and consequently demeans the asset and debases the option premium of return (kills $\mu = pu + qd - 1$) so that it is only accurate for the risky bank account *Y*, i.e., cash with full market risk. In terms of the components of the portfolio,

$$h_0 S_0 - f_0 = vh_u S_u - vf_u = vh_d S_d - vf_d$$

From the delta $h_0 = (f_u - f_d) / (S_0 u - S_0 d)$, rearranging $h_0 S_0 - f_0 = vh_0 S_u - vf_u$ for a premium f_0 of

$$\begin{aligned} f_0 = h_0 S_0 - vh_0 S_u - vf_u &= \left(\frac{f_u - f_d}{u-d}\right)(1-vu) + vf_u = \frac{(f_u - f_d)(1-vu) + v(u-d)f_u}{u-d} = \frac{f_u(1-vd) + f_d(vu-1)}{u-d} \\ &= v\left(\frac{v^{-1}-d}{u-d}\right)f_u + v\left(\frac{u-v^{-1}}{u-d}\right)f_d \\ &= vp'f_u + vq'f_d \end{aligned}$$

where $p' = (v^{-1} - d) / (u - d)$ and $q' = 1 - p' = (u - v^{-1}) / (u - d)$, which is called by risk-neutral valuation the 'risk-neutral-probability measure' which is a complete misnomer and more accurately referred to as the return-neutered (as in a desexed dog) probability measure. The connotation that risk neutral valuation (return-neutered) then draws is that the initial option price is the expected present value of the option, where the expectation is with regard to the two point discrete distribution with the adjusted fake and bogus probability p', for which is inadvertently just a slack variable which forces the constraint of the between node portfolio return fixed flat at the cash rate. It is a misconstrued idea from return-neutered valuation that this is a probability that applies to the dynamics of an asset that has no risk in another world that does not exist; it is, rather, simply an artifact of a modification in the return profile of the market asset and results, ironically, in an asset that has more variance [risk] than the market asset [whilst having no return]. The probability of an upstep for the actual asset is going to be *p* no matter how an option is priced and it must be fully recognized that *p* is exogenous to the pricing of an option. All cashflows of a binomial replicating portfolio zero whenever and however the asset is demeaned and the portfolio debased; it is more correct simply to solve $pu' + qd' = 1 + r$ for u' as then there is **not the scope to grossly misconstrue** the result from a modified parameter that simply demeans the market return. This is particularly so in light of the fact that by debasing the asset by modifying the probability, the **sample space** of the asset (the lattice) does not change but all the statistical attributes do. It must be made very clear, as should be very obvious to any sensible person already, that any one variable cannot have two distributions at the same time regardless of any commonality in sample space. This similarly occurs in continuous time whereby the sample space of a Brownian motion is infinite and dense (continuous) meaning that all Brownian motions have the same sample space; two Brownian motions with differing drifts have identical samples spaces but the allocation of transition probability over the sample space differs whenever the drift rates differ and therefore must be distinct variables. A clear distinction must always be made between the market asset *S* and the demeaned asset *Y* which are two distinct variables that differ for all material purposes, in this context, only in the return characteristics[13]. By the same mathematics, the result from constraining $\Pi_0 = v\Pi_u = v\Pi_d$ is alternatively obtained by any method of nullifying $pu + qd$ including modifying u' and d' instead of p' to take up the slack for the difference i.e., a flat portfolio return at the cash rate occurs whenever Π is debased of market return **however** this occurs. Noting that the total derivative of the premium accurate for *S* is

$$\frac{\Delta f_{j,n}}{\Delta j} = rf_{j,n} - (pu + qd - 1)S_j h_{j,n} + h_{j,n}\frac{\Delta S_j}{\Delta j}$$

so the return-castrated probability measure is obtained instead by setting the return $pu + qd$ equal to the cash rate $1 + r$, which nullifies the market return $puv + qdv - 1$, on delta

[12] a recent value of the **cash rate is 2.5% p.a.** for which it is **doubtful that this exceeds inflation**.

[13] and it is wrong to instead make the distinction by placing subscripts on expectations $E(\cdot)$ for a common variable as though they are the same but somehow different. Two distinct distributions means two distinct variables period.

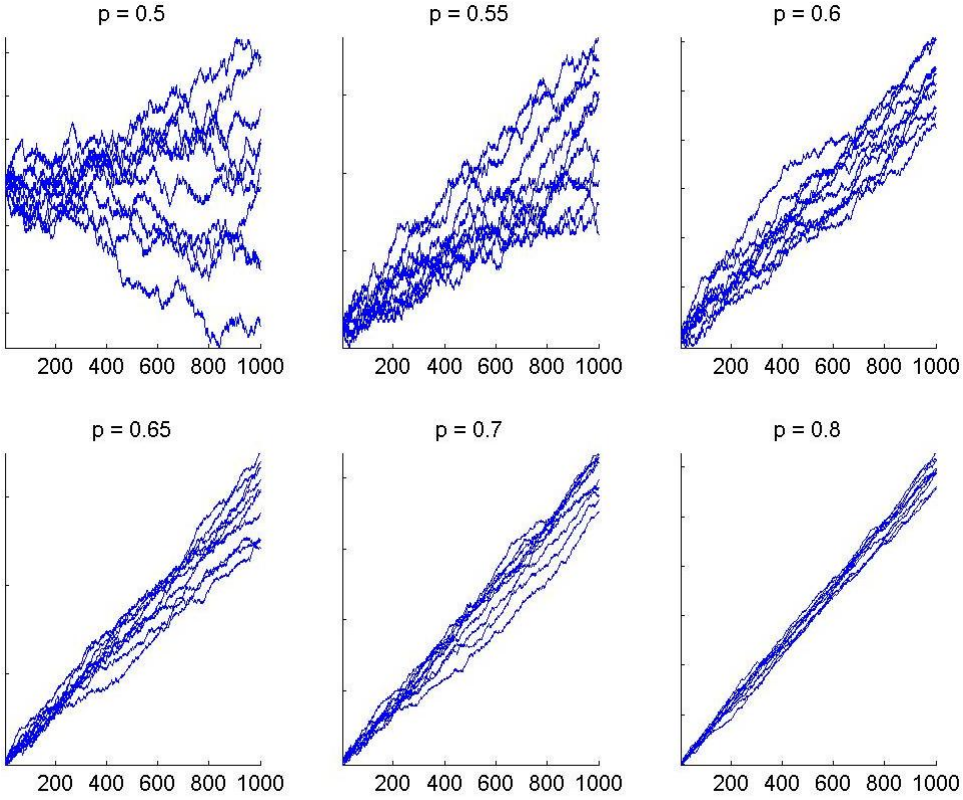
$$\frac{\Delta f_{j,n}}{\Delta j} = rf_{j,n} - rS_j h_{j,n} + h_{j,n}\frac{\Delta S_j}{\Delta j}$$

by solving $p'u + q' = 1 + r$ for p' making the delta side of the total portfolio difference net of the delta update $\Delta\Pi_{j,n} = (p'u + q'd - 1)S_j h_{j,n} - rf_{j,n}$ fixed and flat at *r*

$$\begin{aligned} &p'u + q'd = v^{-1} \\ &p'u + (1-p')d = v^{-1} \\ &p'(u-d) + d = v^{-1} \end{aligned} \qquad \Rightarrow \qquad p' = \frac{v^{-1} - d}{u - d}$$

for which it is clearly seen that risk neutral arrives at this same result by saying that the portfolio earns the cash rate $\Pi_0 = v\Pi_u = v\Pi_d$ and backing out the premium which simply forces this constraint to solve for the solution of $p'u + q'd = 1 + r$ for p' after neglecting the fact that the market asset earns a market rate of return. This is simply a case of converting the coefficient of delta for the return on the market asset *S* to the cash rate by adjusting the probability with the slack variable Δ. The end result is the same; a debased premium priced for the demeaned asset *Y* rather than the market asset to deliver *S*. This is particularly obvious in continuous time where *S* and *Y* differ only in the mean; here in the context of a binomial walk, the variance is larger for the debased asset because of the functional characteristics of the mean and variance properties of the proportional walk as a function of *p*.

The binomial model for a market asset *S* is not suitable to financial securities for modifications of the probability parameter to adjust for [exponential] drift which comprises the market return. Increasing *p* increases return whilst at the same time decreases the volatility. This means that higher return is associated with lower risk! This is contrary to what is consensus about markets and return. 10 simulated paths of returns on *S* (log *S*) for various *p* and consequently for various returns of the market asset is displayed below (from which are exponentiated for the proportional random walk which are so severe that displaying the different graphics is pointless because of the increases in orders of magnitude in exponential growth [and appear as exponential singularities]). Higher *p* is associated with higher return and, as made clear with the simulated paths, is also associated with lower volatility/risk.



Each set of simulated paths above have exactly the same parameters apart from the variations in the parameter *p* (*p* = 0.5, 0.55, 0.6, 0.65, 0.7, and 0.8). There are no low paths, for any *p* > 0.5, since after a couple of steps the

probability of a low path becomes obscenely negligible (0.000…0001). With continuous time limits, this fact is forgotten whereby, unlike the debased binomial asset which also has different variance, the debased proportional Brownian motion differs from that of the market asset only in the exponential drift.

You don't need to be a rocket scientist to make money from an exponential asset at $p = 0.8$, nor do you need an option! If you match volatility for the return-neutered (debased) option nearing $p = 0.5$ whereby volatility is matched to return factors (and $p = 0.5$ demeans the return and here appears like a normal market variable) for the asset to be zero-returning, then the asset for which the option is replicated by, and for which finances the mispricing, is that which is something like $p = 0.8$ when the probability is adjusted back to account for return for the fixed return factors matched to data at $u,d = \exp(\pm\sigma\sqrt{\Delta t})$. The asset is remeaned whilst at the same time the variance is reduced, and then used to source the mispricing in the debased premium and keep the portfolio return flat at cash and void of market return; this is not representative of a market asset for which more return is associated with more risk, not less. So mispricings for an option, you think on $p = 0.5$, is then subsidized by trading a non-existent asset with $p = 0.8$ after matching hypothetical binomial market volatility for the debased premium at $p = 0.5$, i.e., subsidizing the mispricing in the debased option premium from exponential returns earned from an asset materially void of variation.

Only at the backward variables does the demeaned asset, by random variable conditioning, equal the asset itself $Y_0 = S_0$. It diverges the very next instant until the next time it is conditioned, for example, at a subsequent rebalance. Whenever the fake probability of the debased asset is used, this implies that each item within the portfolio are all functions of the demeaned asset and this asset only. These items are **(1)** the demeaned asset itself, **(2)** the unbiased premium for this demeaned asset, and **(3)** the delta of this premium with respect to the demeaned asset. A return back to the original probability by removing the slack means a return to the accurate premium and the replication for the market asset, including premiums, deltas, and assets. By keeping the premium debased with slack, the mispricing in the debased premium kills the return for a high signal-to-noise (SNR) variable composed of all return and, for all material purposes, no variance (since removing the slack from the exponential asset gives it back its mean whilst voiding it of variance). Trading the market asset *S* that is then 'all return' is used to make up the difference between return and no return in the option premium and hence used to naively justify the mispricing in the debased premium. Because, among other reasons, variables in practice are not binomial with an inverse relationship between risk and return, it is not possible to do this [subsidize an option by killing it for flat portfolio return and zero NPV] in practice as has been consistently evidenced and noted in the literature and, as will be discussed, nor is there any point in even trying do so to justify the systematic mispricing of options.

If one takes the fake probability of the demeaned asset p' and multiplies the both numerator and denominator by the initial underlying asset value and then solves for the initial asset value in terms of the asset at the end of the first time step, then one obtains the expected present value of the asset, like that of the option, at the two point discrete distribution with the demeaned probability parameter $p' = (v^{-1} - d)/(u - d)$ which returns only the cash rate

$$
\begin{aligned}
p' &= (v^{-1} - d)/(u - d) \\
&= (S_0 v^{-1} - S_0 d)/(S_0 u - S_0 d) \\
&= (S_0 v^{-1} - S_d)/(S_u - S_d) \\
p'(S_u - S_d) &= (S_0 v^{-1} - S_d) \\
S_0 v^{-1} &= p'(S_u - S_d) + S_d
\end{aligned}
$$

separately from making a point, Y_0 should be multiplied for the numerator and denominator rather than S_0 although any symbol can be utilized, which is statistically a result of conditioning on the backward variable of the demeaned asset *Y*.

and therefore,

$$S_0 = vp'S_u + vq'S_d$$

which is the demeaned asset *Y* and is a fallacious paradox which appears as though *S* is the same as *Y*. It is seen clearly that the variable *Y* has a different distribution from the market asset *S* upon which the option is written and is indicative of the obvious statistical error and mispricing in the premium for the option on *S* which is priced for *Y* by the debased premium and consequently mispriced for *S*. In the case of the lattices of two assets *Y* and *S*, and various functions of these variables, *S* and *Y* are the same at any infinitesimal moment only by conditioning (at the backward variables for which they can, in fact, be anything you like by conditioning as above for $S_0 = Y_0$), and then diverge at the very next instant since they grow at different rates, and in the case of a binomial variable with a fake probability, also vary at different rates. In this case for a changed *p*, the debased variable is still on the same sample space, but **at a different position on the sample space because of the differing likelihoods and differing relative frequency of upsteps**. The distribution of the forward market price is an

exogenous assumption and therefore should not and cannot be modified by the mathematical arguments. The forward market price, given that it is log-binomial with parameter p , will still be log-binomially distributed with parameter p (rather than p') regardless of what one assumes regarding the option or hedge portfolio. It is not at all sensible or in any way valid to say that one variable S has two different distributions. Any one variable cannot ever have two distributions at the same time; this is an absurdity and contradiction. If S is the variable of the asset, then this is the variable upon which the option premium must be based (with the correct mean) if the option is to be correctly priced.

With the return-neutral arguments whereby the asset is debased by changing p, probability is inadvertently confounded with return and time value. In the present value formula $\Sigma p(t)v(t)u(t)$, which is the sum of terms each composed of a probability, a discount factor, and a return factor, the probabilities are confounded with the return and discount factors and is why the fake probability of the debased asset (the "risk-neutral measure") includes negative probabilities and probabilities greater than one; when too much probability has been shifted out of the distribution in slack to demean the return by setting $p'u+q'=1+r$ by varying p' for which the solution returns a result outside the possible values for a probability. In particular, if Δ is the slack variable for p', then debasing the asset by modifying p is equivalent to reducing the upstep probability by Δ and increasing the downstep probability by Δ ($p' = p - \Delta$ and $q' = q + \Delta$) to make **upsteps less likely** and **downsteps more likely** to **kill the market return**. Negative values of the fake probability p' then occur whenever the amount of return Δ taken out of the upstep probability (to offset the market return of S) is larger than p, i.e., whenever the slack in p' is larger than p. Whenever $\Delta > p$, the values of the fake probabilities p' and q' are then both outside the mandatory zero-one range. This is simply an artifact of modifying probability (surreptitiously by stating that $\Pi_0 = v\Pi_u = v\Pi_d$) as a means to modify the return profile of the market asset to demean it and misprice the option, and then with the replication, hope to earn it all back again by doing a whole lot of trading to subsidize the mispricing and, in doing so, kill the option. There is no "alternative world for which some non-existent hypothetical probability distribution applies" but rather just a debased option for which it is hoped to earn back the mispricing by trading the market asset (but failing always) by letting the investment returns or losses earned on trading the market asset accrue to the purchaser of the option rather than the portfolio so that the portfolio always remains void of market return and castrated of the benefits of the financial markets. This is nothing more than a [failed] circumstance of subsidization; a subsidization of a mispriced premium with trading returns earned or incurred in return for flatlining the option. In the case where the trading in the underlying asset is loss making, the purchaser of the option premium subsidizes the investment losses of the portfolio. In simple terms, **(1)** positive excess trading returns from S finances a reduction in the option premium to that priced for Y (the debased premium) or **(2)** an increased option premium to that priced for Y [for a put or an under returning asset] finances trading losses from S. The latter occurs for an overpriced option whenever an excess returning market asset is traded short or an under returning market asset is traded long. Return-neutered arguments also state that every option, both long and short, must be flatlined in order to subsidize every debased premium for every option within the market which means that every option must be flatlined so that the market prices are consistently debased and everyone spends their working careers [when they are not discrediting people] subsidizing everyone else's premiums to flatline all of the option payoff profiles for a nullified and redundant market. In actuality, there is so much sampling variation, as per such a poor and misunderstood replication (and hence completely void replicating arguments) and the theory regarding multinomial walks and unhedged terms, that such an attempt at subsidizing a mispriced option with trading returns, for a flat portfolio return at r and a flatlined option, is completely defunct and irrelevant since it fails always with probability one, as evidenced from the market refusing to be flatlined by the ignorant and misguided theory synonymous with the debasement of derivatives, and instead only consistently evidencing the fact that the market itself has rejected the abysmal theory with things like the volatility skew (which takes up the slack variable Δ for the bias [the mean] in the debased prices and backs it out again via volatility) and observations of the fact that option prices are not debased (Black & Scholes) prices and never have been. The volatility skew is the derivatives market crying out that it does not want to be debased. Out of the context of an offsetting and subsidizing asset, the debased option prices are outright wrong. **Risk-neutral valuation** is more attune with **communism** which **bars profits, private property, and freedom** since risk-neutral valuation attempts to bar returns and the economic value of derivatives as securities derived from the spot markets and is contrary to all acceptable and sensible notions of markets.

Since p is not the only algebraic quantity in the equation resulting from setting $puv+qdv$ to 1 and not the only parameter that can be varied to adjust for slack, it is not the only method of demeaning the underlying asset and debasing the premium. Note that all of the alternative methods are the same in the continuous time limit, namely, a debased option premium with 100% of the market risk and less than 0% of the market return [in deficit of nothing by $\frac{1}{2}\sigma^2$]. The other methods are discussed and involve removing the market return by modifying the return factors of the steps to kill the market return and debase the portfolio for which the relationship between

risk and return is in the same direction. This is the correct direction for what is known about the financial markets. Once the underlying asset is demeaned, whatever the debasement method (whichever parameter is varied for the slack), the between node return components, net of the mandatory rebalance $Y_{j+1}\Delta h_j$, then become flat at the cash rate

$$\begin{aligned}\Pi_{j+1} &= (p'u + q'd)h_j S_j - (1+r) f_j \\ &= (1+r)h_j S_j - (1+r) f_j \\ &= (1+r)\Pi_j\end{aligned}$$

and if $S_{j+1}\Delta h_j$ is used to rebalance, then the mispricing in the debased premium kills the return on the *S* so that it is essentially the same as trading the asset with no return $Y_{j+1}\Delta h_j$. It is clearly seen that p' simply demeans the asset and debases the premium and forces the flat and constant between node return (and zeroes all net cashflows) rather than an interaction between the return on the market asset and the cash rate on the option which is not otherwise flat nor constant because the amount of the market asset contained in the portfolio is always greater than the value of the short option, as is the return[14], and as was just proved in the preceding topics. Therefore, the constraint $\Delta\Pi_j = r\Pi_j$ characterizes debased option replicating portfolios and premiums. So if the 'risk-neutered-probability-measure' differs from the upstep probability of the market asset (i.e., if the market asset has return and the debased probability has non-zero slack $\Delta > 0$), then the option expected value for the premium is a statistically biased and inaccurate expectation and is consequently mispriced to the extent of the bias, the bias being the statistical inaccuracy of the debased premium backed out from $\Delta\Pi_j = r\Pi_j$ in reference to the correct and accurate premium for the market asset to deliver *S*. To the extent that the return of the demeaned asset *Y* differs from that of the market asset *S*, both assets have the same amount of statistical variation (the binomial premium for $p - \Delta$ has more risk than *S* yet no return). The premium based upon the demeaned asset *Y* is accurate only for the demeaned asset. It is biased and inaccurate for all assets other than an oxymoron risky bank account *Y*, including biased and inaccurate for the market asset *S* for which the option expires and must be delivered if the option expires inthemoney.

Recognizing that the asset will only change with respect to the true probability *p*, since it is irrefutably exogenous, the backwards expected present value can alternatively be written as a correct recursive expectation

$$f_0 = vp(p'/p) f_u + vq(q'/q) f_d$$

which shows that the demeaned premium alternatively (and more sensibly) describes an option premium obtained by discounting the option on the actual asset more heavily for upsteps and less heavily for downsteps. This accentuates the statistical bias and inaccuracy for every additional step and so increases the statistical bias and inaccuracy with the number of steps to option expiry, and only at expiry are they ever in parity, i.e., at the time the security ceases to be a derivative and converts to a spot transaction [if it is not turned down]. The **statistical bias and inaccuracy** of the debased price is therefore **an increasing function of term to maturity** and long dated options are therefore massively mispriced; the greater the time to expiry, the greater the mispricing in the debased premium. For excess returning market assets $pu + qd > 1 + r$ for which $(p'/p) < 1$ and $(q'/q) > 1$ (i.e., $v(p'/p) < v_u'$ and $v(q'/q) > v_d'$), options are massively underpriced for long dated call options and massively overpriced for long dated put options. If anyone wants to waste their time further with risk-neutral valuation (with backing out numerical skew in σ), evidence for this for continuous time would be a more pronounced and accentuated volatility skew for long dated options for larger statistical bias and compounded slack that accumulates with extensions in term to maturity. Given that markets are all about return, it is suggested that instead **return-volatility ellipses** are backed out from market prices equated to statistically accurate premiums rather than debased premiums. This is sensible and consistent with Institutional profit motives of those that trade market derivatives, whom would generally not be interested in subsidizing other peoples premiums with hard earned trading returns earned from killing their own option positions.

Another way of viewing what is actually happening is to look at it from the perspective of modified return factors, rather than modified and fake probability, or a variable discount rate. The variable *Y* is then an asset that, instead of increasing to $S_u = uS_0$ if there is an upstep and $S_d = dS_0$ if there is a downstep, changes to $Y_u = \big((p'/p)u\big)Y_0$ if there is an upstep and $Y_d = \big((q'/q)d\big)Y_0$ if there is a downstep when probability is viewed in conjunction with return [without consideration of the exercise price and confounding probability and return which is the major problem with varying *p* in order to demonstrate the changed return profile]. For a fake prob-

[14] for ordinary market conditions.

ability p' for debasement that is smaller than the original $p > p'$ the return factor for the upstep is smaller $(p'/p)u < u$ and the return factor for the downstep is larger $(q'/q)d < d$ which has the effect of demeaning the market asset *S* by making upsteps earn less as well as **downsteps incur less**, for that which then has no return net of cash after averaging the statistical variation. The mean return of the demeaned asset is

$$\begin{aligned} E(Y_1 \mid Y_0) &= \big((p'/p)u\big)pY_0 + \big((q'/q)d\big)qY_0 \\ &= (p'u + q'd)Y_0 \\ &= (1+r)Y_0 \end{aligned}$$

which exactly offsets the discounting factor applied to all cashflows; *n* periods of return for *Y* amounts to v^{-n} on average $E(Y_n \mid Y_0) = (1+r)^n Y_0 = v^{-n}Y_0$ which exactly offsets the *n* period discount factor v^n so the expected present value is completely null $v^n E(Y_n \mid Y_0) = v^n(1+r)^n Y_0 = Y_0$ zeroing all net cashflows resulting from nullifying the option. When inflation and the cash rate are close proxies (say 2.5% p.a.), which is so **in many well documented circumstances [including this very moment as I write this], you may then zero the cash rate and ask yourself if it still makes sense to say that 'it earns nothing at all and therefore must be risk free'!?** For the actual market asset to deliver that returns $pu + qd > 1 + r$ over any step

$$\begin{aligned} E(S_1 \mid S_0) &= pS_u + qS_d \\ &= puS_0 + qdS_0 \\ &= (pu + qd)S_0 \end{aligned}$$

and the expected return for *n* periods is

$$E(S_n \mid S_0) = (pu + qd)^n S_0$$

for which the present value net of cash is

$$v^n E(S_n \mid S_0) = (puv + qdv)^n S_0$$

so the present value of the difference between the two return profiles is then $puv + qdv - 1$ per step which is the non-null market return on *S* and there is a total compounding factor of $(puv + qdv)^n$ in excess of non-performing and redundant assets. For the debased portfolio, whatever the asset earns, it is killed by the mispricing in the option and the replication kills the option. The debased option premium is the premium that is statistically accurate and unbiased for Y, not S, and whenever such securities earn anything different from the cash rate, valuing options with debased prices is the structural mispricing of the risk and return that options on such securities represent, i.e., a structural mispricing of the option on the market asset *S* which must be delivered if the option expires inthemoney [if it is not turned down at the option date]. The debasement of options is an overwhelming impediment which blocks derivatives from reaching their full potential within the context of efficient markets and sensible mathematics.

Backing-out the Dynamics of the Binomial Option Premium

Backing out the backwards recurrence for the statistically accurate premium from a delta neutral option replicating portfolio

A binomial option replicating portfolio is delta-neutralized by taking an offsetting position in the underlying market asset to kill the delta of the option [rip out its heart] so that $\Pi_u = \Pi_d$ net of the rebalance. It is then debased if the change from Π_0 to Π_1 is also held fixed and flat at the cash rate *r* with $v\Pi_1 = \Pi_0$ i.e., $v\Pi_u = v\Pi_d = \Pi_0$, instead of being free to earn the market return $pu + qd - 1$ on the market asset and incur the cash rate *r* on the short option net of the replicating effects of the offsetting delta. Firstly considering a delta-neutral portfolio that has not been debased; since delta is sufficient to kill the variation in a binomial asset or a derivative of a binomial asset, ΔS, a delta neutral portfolio is the same with an upstep as with a downstep as per the delta-neutral condition $\Pi_u = \Pi_d$ so the expectation coincides with the value $E\big(\Pi_1 \mid \Pi_0\big) = \Pi_1$ net of the outstanding delta update. So

$$\begin{aligned}\Pi_1 &= E\left(\Pi_1 \mid \Pi_0\right)\\ &= p\Pi_u + q\Pi_d\\ &= p(S_u h_0 - f_u) + q(S_d h_0 - f_d)\\ &= p(S_0 u h_0 - f_u) + q(S_0 d h_0 - f_d)\\ &= (pu + qd)S_0 h_0 - (pf_u + qf_d)\end{aligned}$$

Alternatively, the return components were proved to be

$$\Pi_1 = (pu + qd)S_0 h_0 - (1+r) f_0$$

by a variety of methods. Then by equating the two expressions for the forward portfolio value (which are identities), $\Pi_1 = (pu + qd)S_0 h_0 - (pf_u + qf_d)$ with $\Pi_1 = (pu + qd)S_0 h_0 - (1+r) f_0$, for

$$(pu + qd)S_0 h_0 - (pf_u + qf_d) = (pu + qd)S_0 h_0 - (1+r) f_0$$

and backing out the backwards recursion in the premium

$$f_0 = vpf_u + vqf_d$$

which is the backwards recursion for the statistically accurate premium for the market asset to deliver and is ordinarily just an input to deduce the portfolio return components. Rather than equating identities and backing out the recurrence, the correct and rigorous method of deducing this equation is by integrating [summing in this discrete context] the backward dynamics of the market asset to transform it to that of the premium as per the approach taken by and original to this thesis[15].

Backing out the backwards recurrence for a debased premium from a debased delta neutral option replicating portfolio

If, instead, the delta-neutral portfolio is debased by saying that it earns the cash rate and the cash rate only $v\Pi_1 = \Pi_0$, then the portfolio return components are held fixed and flat at r

$$\begin{aligned}\Pi_1 &= (1+r)\Pi_0\\ &= (1+r)S_0 h_0 - (1+r) f_0\end{aligned}$$

Backing out the recursion for the backward dynamics of the option premium from the debased portfolio $v\Pi_1 = \Pi_0$ then returns the recursion for the premium that has been debased of market return and void of a mean net of cash.

(i) debasing by varying the parameters for the probabilities *p* and *q* for the slack between return and no return

Debasing by taking slack from p (for $p' = p - \Delta$ and $q' = 1 - p' = 1 - (p - \Delta) = q + \Delta$) to make up the difference between return and no return

$$\begin{aligned}\Pi_1 &= E\left(\Pi_1 \mid \Pi_0\right)\\ &= p'\Pi_u + q'\Pi_d\\ &= (p'u + q'd)S_0 h_0 - (p' f_u + q' f_d)\\ &= (1+r)S_0 h_0 - (p' f_u + q' f_d)\end{aligned}$$

since $p'u + q'd = 1 + r$ with the probabilities adjusted with a (additive) slack variable Δ. Equating the two expressions for the debased portfolio, that of $\Pi_1 = (1+r)S_0 h_0 - (p' f_u + q' f_d)$ with debased parameters (p' and q') with the forward portfolio implied by $v\Pi_1 = \Pi_0$ ($\Pi_1 = (1+r)S_0 h_0 - (1+r) f_0$), results in the identity

$$(1+r)S_0 h_0 - (1+r) f_0 = (1+r)S_0 h_0 - (p' f_u + q' f_d)$$

and backing out equation for the debased premium

$$f_0 = vp' f_u + vq' f_d$$

[15] for which it became definitively clear to me from research and mathematics in 2010, after trying to write around the theory of debased option portfolios, that premiums void of return are not in any way correct.

which is the backwards recursion for the debased premium for which the offsetting asset kills the option and the mispricing in the debased premium kills the market return earned on flatlining the option, holding the portfolio return fixed and flat at the cash rate over the replication period voiding all cashflows of market return.

(ii) debasing by varying the parameters for the return factors *u* and *d* for the slack between return and no return

Debasing by taking slack from u ($u' = u / \Delta$ and $d' = d / \Delta \Rightarrow \Delta = (puv + qdv)$) to make up the difference between return and no return

$$\begin{aligned}\Pi_1 &= E\left(\Pi_1 \,|\, \Pi_0\right)\\ &= p\Pi_{u'} + q\Pi_{d'}\\ &= (pu' + qd')S_0h_0 - (pf_{u'} + qf_{d'})\\ &= (1+r)S_0h_0 - (pf_{u'} + qf_{d'})\end{aligned}$$

since $pu' + qd' = 1 + r$ with the return factors adjusted with a (proportional) slack variable Δ. Exactly the same as that with a debased portfolio from probability adjusted for slack; equating the two expressions for the debased portfolio, that of $\Pi_1 = (1+r)S_0h_0 - (pf_{u'} + qf_{d'})$ with the debased parameters (u' and d') with that of $v\Pi_1 = \Pi_0$ (i.e., with $\Pi_1 = (1+r)S_0h_0 - (1+r)f_0$), results in the identity

$$(1+r)S_0h_0 - (1+r)f_0 = (1+r)S_0h_0 - (pf_{u'} + qf_{d'})$$

and backing out the equation of the debased premium

$$f_0 = vpf_{u'} + vqf_{d'}$$

which differs from that with debased slack in probability only in that the subscripts are primed (with slack Δ) rather than the coefficients; this is the backwards recursion for the debased premium for which the offsetting asset kills the option and the mispricing in the debased premium kills the market return earned on flatlining the option, holding the portfolio return fixed and flat at the cash rate over the replication period. This is entirely analogous to that for which the return is debased via the probability parameters and entirely analogous to that in continuous time for the PDE of the debased price for an option on proportional Brownian motion such as that of Black and Scholes (1973).

The Debased Option Premium and Deltas

A market asset can be debased and zeroed of return by letting any one of the parameters in $puv + qdv = 1$ vary for the slack Δ between return and no return, i.e., to take up the slack between market return $pu + qd$ and cash $1 + r$. There are then either probabilities, return factors, or the discount factor that can be varied to convert $pu + qd$ to $1 + r$. Since p and q are complementary probabilities, these have one degree of freedom in being varied to debase the asset. Subject to the constraint that $p = 1 - q$, the method [implicitly or explicitly] is to deduct slack Δ from upstep probability to make upsteps less likely and add slack Δ to downstep probability to make downsteps more likely. With a restriction on the return factors that they are reciprocal, $u = 1 / d$, there is similarly one degree of freedom to be varied to debase the asset by changing the return ratios. Upsteps ratios are made proportionately smaller $u' = u / \Delta$ and downsteps ratios are made proportionately larger $d' = d\Delta$ so that the asset earns less on upsteps and incurs less on downsteps. Since the restriction that the return ratios are proportional is not mandatory, if this is relaxed, there are then half a dozen more methods to debase the asset by modifying the return factors but there is one that stands out from the crowd, and this is that for which kills the market return directly from the walk variable and does so independently of the basis function and step generator. This is whereby the slack variable Δ is used to reduce both upsteps and downsteps so that upstep and downstep ratios are made proportionately smaller, $u' = u / \Delta$ and $d' = d / \Delta$, so that the asset earns less on upsteps and incurs more on downsteps. This is the direct walk analogue to that in continuous time and is the preferred method amongst the alternatives. This does not dispense with the original return factors, but simply places a sink directly on the market asset; and converts the risky market asset to risky cash [an oxymoron risky bank account proper]. This does not keep the walk symmetrical on logarithms but it does leave the walk variance unchanged on logarithms which is ideal for comparisons with continuous time as it leaves variations in the mean to debase the asset independent from volatility in the second cumulant.

Again, all of these methods zero all net cashflows and result in the debased premium of proportional Brownian motion [or degenerate to a pseudo-option on the discount factor] upon taking continuous time limits. The addi-

tional methods of debasing via the return ratios (but not probabilities) will be considered as "Other methods of demeaning the market asset *S*" at the end of this topic, as it is much more theoretically sensible to change the return factors than to change exogenous probabilities.

So there are two distinct approaches by which to demean the market asset and debase the premium, each by setting the mean effective return $pu + pd - 1$ equal to the cash rate r so that the portfolio return is flat at the cash rate and independent of the size of delta due to the overall return from the items in the debased portfolio factoring from being fixed flat. The two approaches to demean the asset are those obtained by either taking slack from the probability (subject to $p = 1 - q$) or to take slack from the return ratios, i.e.,

(1) varying the probability of an upstep, p (making upsteps less likely and downsteps more likely); or,
(2) varying the extent to which each step grows, u (making upsteps earn less and downsteps incur more);

which both debase the premium by changing the return profile of the underlying asset. The slack then turns up again in implied volatility. Differing features of these methods include **(A)** the mean-variance relationship, **(B)** the distribution-sample-space (lattice) relationship, and **(C)** the remaining parameter that is matched to market data;

With regard to **(A)** the mean variance relationship, the former **(1)** (decreasing p) increases the variance when debased with slack and the latter **(2)** (decreasing u) decreases the variance when debased with slack, the former a negative relationship and the latter a positive relationship between risk and return;

With regard to **(B)** the distribution-sample-space (lattice) relationship, the former **(1)** (varying p) changes the probability distribution for an unchanged lattice of the sample space while the latter **(2)** (varying u) changes the lattice of the sample space (from returns) for an unchanged probability distribution over the lattice;

With regard to **(C)** the parameter that is matched, the former **(1)** dispenses with the original probability and hence matches the return factors to market data while the latter **(2)** dispenses with the original return factors if the step ratios are kept in proportion and matches the probability parameter to market data, whereas **slack from both upsteps and downsteps keeps the various features fully parameterized.**

The bias in the debased price is slightly smaller for that debased by taking slack out of probability since the demeaned asset, whilst having no market return, has more than 100% of the market risk; the lower mean makes the debased call premium smaller whilst the increase in variance offsets this [and the bias] by making it slightly larger. That which takes slack from upsteps and puts slack into downsteps $u' = u/\Delta$ and $d' = d\Delta$ is slightly smaller again than that which takes slack from both upsteps and downsteps $u' = u/\Delta$ and $d' = d/\Delta$ ($\Delta = (puv + qdv)$). Each of these premiums limit to the continuous time debased premium of Black and Scholes (1973).

Although modifying the sample space results in some difficulties from the changed lattice (an indicator of further problems with attempting to flatline options), modifying u shows clearly what is happening and is better because then higher return is associated with higher risk. Noting that, in continuous time, the variance is not varied by demeaning the asset and debasing the option premium. So, in continuous time as a limit of the walk by making the step size small, the difference between the two methods is o(dt) both in numerical and theoretical terms as there is only the drift of the geometric Brownian motion to adjust to produce the zero returning asset Y (the variance does not change when the portfolio is debased) and there is no difference between sample spaces since each is continuous (dense by making the step size small and the number of rebalances infinite) over the infinite range of the positive real line. The net effect in continuous time, therefore, is the same; namely, a debased option void of market return μ for which it is hoped to earn back the mispricing by flatlining the option; otherwise, the option is mispriced outright by the debased premium.

(1) Demeaning the market asset *S* by modifying probability

It is usual, in fact all of the prior and current literature [regarding debased options], takes the approach to modify the probability of an upstep to demean the market asset of return in order to systematically misprice options. This literature refers to the distribution of the demeaned asset Y as the risk-neutral measure. More suitable terminology for this distribution is the return-neutered measure. Solving $p'u + q'd = v^{-1}$ for p' for the modified parameter of the step generator, for flat coefficients in the portfolio return components for a flat debased portfolio return, results in

$$p' = \frac{v^{-1} - d}{u - d}$$

for which is an inadvertent method of simply swapping $pu + qd - 1$ and r in the coefficient of the option delta in the dynamics of the premium. This is just a method of debasing the underlying asset by varying the probability parameter p with an additive slack variable Δ by taking Δ out of the upstep probability **to make upsteps less likely** and adding Δ to the downstep probability **to make downsteps more likely** for $p' = p - \Delta$ and $q' = q + \Delta$ so that the asset is zero returning net of the cash rate $p'u + q'd = 1 + r$. Solving for the slack Δ

$$u(p - \Delta) + d(q + \Delta) = 1 + r$$
$$(pu + qd) - (u - d)\Delta = 1 + r$$

i.e.,

$$\Delta = \frac{pu + qd - 1 - r}{u - d}$$

so that

$$p - \Delta = p - (pu + qd - 1 - r)(u - d) = (pu - pd - pu - qd + 1 + r)/(u - d) = (1 + r - d)/(u - d) = p'$$
$$q + \Delta = q + (pu + qd - 1 - r)(u - d) = (qu - qd + pu + qd - 1 - r)/(u - d) = (u - 1 - r)/(u - d) = q'$$

which is the same as that by debasing by simply solving $p'u + q'd = 1 + r$ for p' and q'

$$p' = (1 + r - d)/(u - d) \qquad\qquad q' = (u - 1 - r)/(u - d)$$

since p and q are binomial complements so that slack taken from p is then added to q. Modifying the probability changes the probability distribution over an unchanged sample space and it must not be misconstrued that because the sample space is unchanged that the two variables are the same. A different distribution and likelihood necessarily means a different variable, entirely independent of any commonality in the sample space. From the mean and variance of a step of the walk, $pu + qd$ and $pq(u - d)^2$ it is seen that unlike the return ratios, p and q are multiplicative in the variance. There is therefore an inverse relationship between the mean and variance for variations in p on either side of $p = 0.5$ as a result of a concave parabola in variance vs p. Among other problems, apart from not leaving the variance unchanged and the relationship being in the wrong direction, it makes absolutely no sense to change probabilities of events and samples spaces that you have absolutely no control over, notionally or otherwise.

Adjusting the parameter p can be viewed in the context of letting the discount factor on the option be a perfect correlate with the asset. This then, in effect, is a method of modifying the cash rate discount factor applied to the option premium [but not the other cashflows] to sink upstep returns or source downstep losses. Modifying the discount factor that is a return modified version of S, $v_u' = v\,p'/p$ and $v_d' = v\,q'/q$, is algebraically the same as modifying p although is conceptually more sensible since there is less scope to misconstrue the results and extrapolate nonsense; letting the discount factor on the option be a perfect correlate with the asset so that $v_u' = v\,p'/p$ and $v_d' = v\,q'/q$

$$v_u' = vp'/p = v(v^{-1} - d)/(u - d) = (1 - vd)/(u - d) < v$$
$$v_d' = vq'/q = v(u - v^{-1})/(u - d) = (vu - 1)/(u - d) > v$$

which then sinks (takes) returns on upsteps and sources (puts back) returns on downsteps [on the option] for flat market return at cash which then debases the premium and zeroes all net cashflows. The return profile of the underlying asset is then converted to risky cash, in the case here, to debase it of return in order to misprice the option in the hope of earning it (the mispricing) all back again with the replication that flatlines the option. This is what risk-neutral valuation has misconstrued as "being true in every world", viz., subsidizing the mispriced premium of the killed option with the trading returns earned on flatlining it.

The debased call and put option premia are those which are statistically unbiased and accurate for the demeaned asset Y with the fake probability p'. So, from the statistically accurate premium with the correct probability for the market asset S, the debased call option premium is then that with the probability adjusted with the slack variable $p' = p - \Delta$

$$\begin{aligned}
f_{0,n} &= S_0 v^n \sum_{x=a}^{n} {}^nC_x (p'u)^x (q'd)^{n-x} - kv^n \sum_{x=k^*}^{n} {}^nC_x (p')^x (q')^{n-x} \\
&= S_0 (p'u + q'd)^n v^n \sum_{x=a}^{n} {}^nC_x \left(\frac{p'u}{p'u + q'd} \right)^x \left(\frac{q'd}{p'u + q'd} \right)^{n-x} - kv^n \sum_{x=a}^{n} {}^nC_x (p')^x (q')^{n-x} \\
&= S_0 \sum_{x=a}^{n} {}^nC_x (vp'u)^x (vq'd)^{n-x} - kv^n \sum_{x=a}^{n} {}^nC_x (p')^x (q')^{n-x} \\
&= S_0 \bar{F}_{B(n,vp'u)}(a) - kv^n \bar{F}_{B(n,p')}(a)
\end{aligned}$$

since $(p'u + q'd)^n v^n = 1$ for null return net of cash time value and where $a = \log(k / S_0 d^n) / \log(u/d)$. For a put, the debased premium is

$$\begin{aligned}
f_{0,n} &= kv^n \sum_{x=0}^{a} {}^nC_x (p')^x (q')^{n-x} - S_0 v^n \sum_{x=0}^{a} {}^nC_x (p'u)^x (q'd)^{n-x} \\
&= kv^n \sum_{x=0}^{a} {}^nC_x (p')^x (q')^{n-x} - S_0 (p'u + q'd) v^n \sum_{x=0}^{a} {}^nC_x \left(\frac{p'u}{p'u + q'd} \right)^x \left(\frac{q'd}{p'u + q'd} \right)^{n-x} \\
&= kv^n \sum_{x=0}^{a} {}^nC_x (p')^x (q')^{n-x} - S_0 \sum_{x=0}^{a} {}^nC_x (vp'u)^x (vq'd)^{n-x} \\
&= kv^n F_{B(n,p')}(a) - S_0 F_{B(n,vp'u)}(a)
\end{aligned}$$

similarly since $(p'u + q'd)^n v^n = 1$ and where $a = \log(k / S_0 d^n) / \log(u/d)$. The delta of the debased call option is

$$\begin{aligned}
h_{0,n} &= \sum_{x=a}^{n-1} {}^{n-1}C_x (vp'u)^x (vq'd)^{n-1-x} \\
&= \bar{F}_{B(n-1,vp'u)}(a)
\end{aligned}$$

and for the debased put option,

$$\begin{aligned}
h_{0,n} &= -\sum_{x=0}^{a-1} {}^{n-1}C_x (vp'u)^x (vq'd)^{n-1-x} \\
&= -F_{B(n-1,vp'u)}(a-1)
\end{aligned}$$

for the exercise price on the walk sample space at the forward variables. If k is not on the samples space of the forward transition variable, there is the additional term at $x = a - 1$ for the call option delta and $x = a$ for the put option delta.

One of several irrefutable problems with modifying the probabilities to debase an asset is that the modified probabilities include negative probabilities and probabilities greater than one whenever the slack to reduce up-step probabilities $\Delta = (pu + qd - 1 - r)/(u - d)$ exceeds p, in which case the binomial CMFs are not technically defined. In effect, this results from too much return being removed from probability in the underlying asset to debase it and the resulting option premium. This is just another item of irrefutable evidence that the "risk-neutral probability measure" is not a valid probability distribution, not for modifying the return profile, and does not happen when it is return factors that are varied with slack to debase the underlying asset and resulting premium.

Solving the recurrence for the option PREMIUM debased by varying the parameter *p*

The recursion for the debased premium is the recursion for the accurate premium but instead with the debased probability parameter p' subject to the boundary condition for the debased option accurate for the oxymoron risky bank account Y. Conditional on the spot price $Y_0 = S_0$, these are

$f_0 = v^n \max(Y_n - k, 0 \mid S_0) = v^n \max(S_0 u^X d^{n-X} - k, 0 \mid S_0)$ with $X \sim B(n, p')$ for a call option and

$f_0 = v^n \max(k - Y_n, 0 \mid S_0) = v^n \max(k - S_0 u^X d^{n-X}, 0 \mid S_0)$ with $X \sim B(n, p')$ for a put option

where $p' = p - \Delta$ and $\Delta = (pu + qd - 1 - r)/(u - d)$. From the backwards recursion of the premium debased by taking slack from p

$$f_0 = vp' f_u + vq' f_d$$

In one steps time this is either

$f_u = vp' f_{uu} + vq' f_{du}$ if the notional debased spot increases to $Y_u = S_0 u$ (with frequency p') or

$f_d = vp' f_{ud} + vq' f_{dd}$ if the notional debased spot decreases to $Y_d = S_0 d$ (with frequency q') or

and similarly,

$f_{uu} = vp' f_{uuu} + vq' f_{uud}$ if the notional debased spot increases to $Y_{uu} = S_0 u^2$ (with frequency $(p')^2$) or
$f_{ud} = vp' f_{udu} + vq' f_{udd}$ if the notional debased spot moves to $Y_{ud} = S_0 ud$ (with frequency $2p'q'$) or
$f_{dd} = vp' f_{ddu} + vq' f_{ddd}$ if the notional debased spot decreases to $Y_{dd} = S_0 d^2$ (with frequency $(q')^2$)

Unfolding the recursion for the debased premium f_0 to the boundary at the forward variables at step *n*

$$\begin{aligned}
f_0 &= v(p') f_u + v(q') f_d \\
&= vp'\left(vp' f_{uu} + vq' f_{ud}\right) + vq'\left(vp' f_{ud} + vq' f_{dd}\right) \\
&= v^2 (p')^2 f_{uu} + 2v^2 (p')(q') f_{ud} + v^2 (q')^2 f_{dd} \\
&= v^2 (p')^2 \left(p' f_{uuu} + q' f_{uud}\right) + 2v^2 p'q'\left(p' f_{udu} + q' f_{udd}\right) + v^2 (q')^2 \left(p' f_{ddu} + q' f_{ddd}\right) \\
&= v^3 (p')^3 f_{uuu} + 3v^3 (p')^2 (q') f_{uud} + 3v^3 (p')(q')^2 f_{udd} + v^3 (q')^3 f_{ddd} \\
&\ldots \\
&= v^n \sum_{x=0}^{n} {}^nC_x (p')^x (q')^{n-x} f_{u^x d^{n-x}}
\end{aligned}$$

Since the boundary conditions for the call option with the debased probability is $\max\left(Y_n - k, 0 \mid S_0\right) = \max\left(S_0 u^X d^{n-X} - k, 0 \mid S_0\right)$ for $X \sim B(n, p')$, the call option premium at the boundary for *x* upsteps is $f_{u^x d^{n-x}} = \max\left(S_0 u^x d^{n-x} - k, 0\right)$ so that

$$\begin{aligned}
f_0 &= v^n \sum_{x=0}^{n} {}^nC_x (p')^x (q')^{n-x} f_{u^x d^{n-x}} \\
&= v^n \sum_{x=0}^{n} {}^nC_x (p')^x (q')^{n-x} \max\left(S_0 u^x d^{n-x} - k, 0\right) \\
&= \bar{F}_{B(n,p')}(a) - kv^n \bar{F}_{B(n,p')}(a)
\end{aligned}$$

where $(p'uv + q'dv)^n = 1$. The put option premium at the boundary for *x* upsteps is $f_{u^x d^{n-x}} = \max\left(k - S_0 u^x d^{n-x}, 0\right)$ since the boundary condition for the put option is $\max\left(k - Y_n, 0 \mid S_0\right) = \max\left(k - S_0 u^x d^{n-x}, 0 \mid S_0\right)$ for $X \sim B(n, p')$ so that

$$\begin{aligned}
f_0 &= v^n \sum_{x=0}^{n} {}^nC_x (p')^x (q')^{n-x} f_{u^x d^{n-x}} \\
&= v^n \sum_{x=0}^{n} {}^nC_x (p')^x (q')^{n-x} \max\left(k - S_0 u^x d^{n-x}, 0\right) \\
&= kv^n F_{B(n,p')}(a) - F_{B(n,p')}(a)
\end{aligned}$$

where again $(p'uv + q'dv)^n = 1$ as a result of the asset [for which the option is priced] and premium being debased.

Solving the recurrence for the DELTA of the option debased by varying the parameter *p*

Solving the backwards recursion for the delta of the debased option by unfolding it over the backward variables to the forward variables at option expiration; from $h_0 = vp'uh_u + vq'dh_d$, in one steps time this either

$h_u = vp'uh_{uu} + vq'dh_{ud}$ if the notional debased spot increases to $Y_u = S_0 u$ (with frequency p') or
$h_d = vp'uh_{du} + vq'dh_{dd}$ if the notional debased spot decreases to $Y_d = S_0 d$ (with frequency q')

Similarly, in two steps time the recursion becomes

$h_{uu} = vp'uh_{uuu} + vq'dh_{uud}$ if the notional debased spot increases to $Y_{uu} = S_0 uu$ (with frequency $(p')^2$)
$h_{ud} = vp'uh_{udu} + vq'dh_{udd}$ if the notional debased spot moves to $Y_{ud} = S_0 ud$ (with frequency $2p'q'$)
$h_{dd} = vp'uh_{ddu} + vq'dh_{ddd}$ if the notional debased spot decreases to $Y_{dd} = S_0 dd$ (with frequency $(q')^2$)

and so forth. Unfolding the recurrence to the forward variables

$$
\begin{aligned}
h_0 &= vp'uh_u + vq'dh_d \\
&= vp'u\left(vp'uh_{uu} + vq'dh_{ud}\right) + vq'd\left(vp'uh_{ud} + vq'dh_{dd}\right) \\
&= v^2(p'u)^2 h_{uu} + 2v^2(p'u)(q'd)h_{ud} + v^2(q'd)^2 h_{dd} \\
&= v^2(p'u)^2\left(p'uh_{uuu} + q'dh_{uud}\right) + 2v^2(p'u)(q'd)\left(p'uh_{udu} + q'dh_{udd}\right) + v^2(q'd)^2\left(p'uh_{ddu} + q'dh_{ddd}\right) \\
&= v^3(p'u)^3 h_{uuu} + 3v^3(p'u)^2(q'd)h_{uud} + 3v^3(p'u)(q'd)^2 h_{udd} + v^3(q'd)^3 h_{ddd} \\
&\ldots \\
&= \sum_{x=0}^{n-1} {}^{n-1}C_x (vp'u)^x (vq'd)^{n-1-x}\, h_{u^x d^{n-1-x}}
\end{aligned}
$$

At the boundary at the forward variables, the delta is one for that part of the range where the option is inthemoney and zero for that part where the option is outofthemoney. Given that the exercise price is on the lattice

$$
h_{u^x d^{n-1-x}} = \begin{cases} 1 & for\ x > a-1 \\ 0 & for\ x < a-1 \end{cases}
$$

where $a = \log(k/S_0 d^n)/\log(u/d)$ so that

$$
\begin{aligned}
h_0 &= \sum_{x=a}^{n-1} {}^{n-1}C_x (vp'u)^x (vq'd)^{n-1-x} \\
&= \bar{F}_{B(n-1,vp'u)}(a)
\end{aligned}
$$

where the term for $x = a-1$ zeroes because of the exercise price being on the lattice. If it is not on the lattice, then there is the additional term added to the delta $(S_0 u^a d^{n-a} - k)\,{}^{n-1}C_{a-1}(vp')^{a-1}(vq'')^{n-a}/S_0(u-d)$ for that corresponding to the term for $x = a-1$ in the summation of the solution to the recursion. For a put option delta, with the exercise price on the lattice

$$
h_{u^x d^{n-1-x}} = \begin{cases} 0 & for\ x > a \\ -1 & for\ x < a \end{cases}
$$

where $a = \log(k/S_0 d^n)/\log(u/d)$ so that

$$
\begin{aligned}
h_0 &= -\sum_{x=0}^{a-1} {}^{n-1}C_x (vp'u)^x (vq'd)^{n-1-x} \\
&= -F_{B(n-1,vp'u)}(a-1)
\end{aligned}
$$

where similarly to the call option delta, the term for $x = a$ zeros because the exercise price is on the lattice. If it is not, then there is the additional term to the delta $-(k - S_0 u^a d^{n-a})\,{}^{n-1}C_a(vp')^a(vq')^{n-a-1}/S_0(u-d)$ for $x = a$ within the summation.

Although there is some nice mathematics, since after debasing the option portfolio with $\Pi_0 = v\Pi_1$ the mathematics is much simplified, distributions with negative probabilities such as those referred to as risk-neutral measures are not in any way valid and it is completely idiotic to try to say that it is because they only exist in some alternative world that does not exist and that the measure (random variable) has two distributions at the same time, one in the real world and one in the non-existent world. This is the equivalent in probability, as a measure on *S* extended from *R*, as saying that a man is both 6 feet tall and 2 feet tall at the same time. They are either two separate men, or someone is spinning nonsense [either way, you can be sure to be suspicious of whoever is claiming to have done the measuring]. It is feasible in continuous time that two Brownian motions with differing drifts are perfectly correlated, for example, a Brownian motion and that same Brownian motion with a sink. In discrete time, a binomial variable that upticks (lands heads) cannot be perfectly correlated with another binomial variable that upticks with a different probability since the relative frequencies differ and **a perfect correlate is defined as all frequencies being the same**. A perfect correlate can only be obtained from one variable via a shift or change of scale, in the context here, with a variation in the return factors. Even then, the variables are necessarily distinct. The only sensible method of demeaning the underlying asset is to modify the return factors and even this is pointless. It is the purpose of this book only to show clearly what is going on and to make clear that anything [such as a paper] or anyone claiming 'one variable with two distributions' is complete nonsense (and an inept charlatan) and this sort of ineptness should not be let anywhere near the management of de-

rivatives. It is no wonder that so many things have gone wrong in derivatives. **Liars[16] should be kept away from derivatives and finance.**

(2) Demeaning the market asset *S* by modifying return factors

Increasing *p* results in higher return whilst at the same time lower volatility. On the other hand, increasing *u* results in higher mean return as well as higher volatility. It is this latter direction which is the correct direction for what is known about the financial markets. Therefore, varying the parameter *u* is the better and only sensible approach to demean the asset since it is consistent with markets for which higher return is associated with higher risk. This is seen simply by looking at the mean $pu+qd$ and variance $pq(u-d)^2$ of a single step of the walk (the per period return and volatility) for which it can be seen that increases in *u* result in increases in both $pu+qd$ and $pq(u-d)^2$ consistent with the notion that more return is associated with more risk.

Debasing by a proportional factor Δ that decreases both upsteps and downsteps with $u'=u/\Delta$ and $d'=d/\Delta$ so that upsteps earn less and downsteps incur more $pu'+qd'=1+r$

$$pu/\Delta+qd/\Delta=1+r$$
$$pu+qd=(1+r)\Delta$$
$$\Delta=puv+qdv$$

which discounts all step ratios, and reduces all steps of the forward transition of market return that is being debased from the asset and premium to zero the cashflows of the replication. So, similarly to any of the other alternatives, **debases the asset and premium and zeroes all net cashflows.** Since the market asset is defined as

$$S_n=S_0u^Xd^{n-X}$$

the debased asset is then

$$\begin{aligned}Y_n&=S_0(u')^X(d')^{n-X}\\&=S_0\left(u/(puv+qdv)\right)^X\left(d/(puv+qdv)\right)^{n-X}\\&=S_0u^Xd^{n-X}/(puv+qdv)^n\end{aligned}$$

and so amounts to simply placing a sink on the market asset to debase it of market return

$$Y_n=S_n/(puv+qdv)^n$$

i.e., **sinks the market return directly from the walk variable and independently of the binomial basis function.** This is how the portfolio and premium is debased in continuous time, whereby the slack is deducted directly from the exponential drift, here the logarithm of the slack is deducted from both the logarithm of the upstep and the logarithm of the downstep

$$\log u'=\log u-\log\Delta \qquad \text{and} \qquad \log d'=\log d-\log\Delta$$

and is simply a linear shift on logarithms

$$\log Y_n=\log S_n-n\log\Delta$$

in proportion to the forward transition length *n*, one deduction of $\log\Delta$ for each forward step. Unlike the methods of debasing via the parameter *p* or via the parameter *u* whilst keeping up and down steps reciprocal, this method does not throw away any parameter; the walk and the slack variable remain dependent upon all parameters of the walk. Although debasing a proportional walk by this method also increases variance

$$Var(Y_n)=Var(S_n)/(puv+qdv)^{2n}$$

, on logarithms, the slack variable does not affect the market variables variance $pq\left(\log u-\log d\right)^2$ since the proportional slack taken from both upsteps and downsteps then cancel in the variance on logarithms for the return variable

$$pq\left(\log u'-\log d'\right)^2=pq\left(\log u/\Delta-\log d/\Delta\right)^2=pq\left(\log u-\log\Delta-\log d+\log\Delta\right)^2=pq\left(\log u-\log d\right)^2$$

[16] which is why finance is corrupt, because risk neutral assholes are such fucking liars.

as per a linear shift on logarithms and hence

$$Var(\log Y_n) = Var(\log S_n)$$

With limits, this is the same as taking $\Delta = \mu + \frac{1}{2}\sigma^2 - r$ directly from μ to leave a drift of $r - \frac{1}{2}\sigma^2$, i.e., reduces all basis components on logs (for returns) by a flat rate Δ while leaving the [infinitesimal log] variance σ^2 unchanged. There is nothing magical about debasing a Brownian motion, all that changes is that the mean [market return] is killed with a slack variable as a result of debasing the portfolio by holding the portfolio return flat at $d\Pi = r\Pi dt$.

The debased parameters u' and d' are defined for all values of p and q, and the debased variable then remains a shifted version of the market asset and hence upticks with the same probability and frequency of the market asset; whenever the market asset upticks so too does the debased asset and whenever the market asset downticks so too does the debased asset. The only difference is that when the debased asset upticks it does so with smaller steps than the market asset and when it downticks it does so with larger steps than the market asset. This then kills the market return.

So, the demeaned asset Y upticks with the same **probability and frequency** as the underlying asset and whenever an uptick of the demeaned asset occurs, an uptick in the market asset S also occurs but the magnitude of the proportional movements differ in order to leave it void of market return. This is in contrast to a modified asset that upticks less often than the original asset and downticks more often, which can only be assessed ex-post in terms of relative frequency, and means that the two variables cannot remain correlated. Modifying the return factors is a sensible interpretation of the demeaned asset upon which the debased option is accurate and the only sensible method of demeaning a market asset. It shows much more clearly that the return is being debased, as a result of the constraint on the portfolio $\Delta\Pi_{j,n} = r\Pi_{j,n}$, and that the premiums are those simply without any mean net of cash. This method changes the sample space but leaves the probability distribution of the lattice unchanged (which is sensible). The resulting debased premium **with** $u' = u/\Delta$ **and** $d' = d/\Delta \Rightarrow \Delta = (puv + qdv)$ is slightly different because of the changed sample space rather than changed distribution over the sample space; for most options, just a couple of cents (percentage points relative) difference **(0.1% difference).**

The analysis proceeds slightly differently although it still results in everything being killed - option, returns, and net cashflows; the returns killed from debasing the portfolio and the premium and option killed from the replication. The difference in approach is that the debased parameters affect the sample space of the forward transition rather than the probability over the sample space of the forward transition. This is much more sensible.

The portfolio is required to trade in the zero-returning asset, but as this asset does not exist in actuality, the portfolio trades in the actual underlying asset and whenever an upstep u occurs the excess growth over that suggested by the modified growth factor $u' = u/(puv + qdv)$ has to be removed or siphoned away, and similarly, whenever a downstep occurs the difference between the change in d and in $d' = d/(puv + qdv)$ has to similarly be removed or siphoned away. Within the context of a replicating portfolio and trading in an offsetting asset, it is the market asset that attempts to earn back all of the return that is notionally demeaned from the underlying asset and resulting debased option, i.e., which attempts to subsidize the mispricing in the debased premium as a rationale and justification to systematically misprice options. So in the context of a replicating offsetting asset, the removed return from the premium is replaced again (sourced) by return of the actual asset which sources the mispricing by flatlining the payoff profile. If there is no replication, for example if you want the option, there is no means by which to earn back the mispricing and the debased premium is outright wrong.

For any method of debasing the premium by modifying the return factors u and d to u' and d', the formula for the debased call option premium is obtained from that for which is accurate for the market asset to deliver, similarly to debasing by changing p, by inserting the debased return ratios into the premium so that it is the accurate premium for the demeaned asset Y rather than the market asset S. For a call option, the debased premium with the return factors adjusted for slack (for either $u' = u/\Delta$ **and** $d' = d/\Delta$ **with** $\Delta = (puv + qdv)$ **or** $u' = u/\Delta$ and $d' = d\Delta$) is

$$\begin{aligned}
f_{0,n} &= S_0 v^n \sum_{x=a'}^{n} {}^nC_x (pu')^x (qd')^{n-x} - kv^n \sum_{x=a'}^{n} {}^nC_x p^x q^{n-x} \\
&= S_0 (pu'+qd')^n v^n \sum_{x=a'}^{n} {}^nC_x \left(\frac{pu'}{pu'+qd'}\right)^x \left(\frac{qd'}{pu'+qd'}\right)^{n-x} - kv^n \sum_{x=a'}^{n} {}^nC_x p^x q^{n-x} \\
&= S_0 \sum_{x=a'}^{n} {}^nC_x (vpu')^x (vqd')^{n-x} - kv^n \sum_{x=a'}^{n} {}^nC_x p^x q^{n-x} \\
&= S_0 \bar{F}_{B(n,vpu')}(a') - kv^n \bar{F}_{B(n,p)}(a')
\end{aligned}$$

since $(pu'v+qd'v)^n = 1$ due to the debasement of the asset and entirely similarly to that for which the portfolio is debased via the probabilities *p* and *q* rather than *u* and *d*. In contrast to that debased via relative frequency, the standardized exercise price also depends on debased parameters $a' = \log\left(k / S_0 (d')^n\right) / \log(u'/d')$ which amounts to the bias in probability in the argument of the CMFs rather than via the parameter *p* of the step generator. This causes problems with the arguments regarding sourcing the mispricing in the debased premium with trading returns earned on flatlining the option. The debased put option premium is

$$\begin{aligned}
f_{0,n} &= kv^n \sum_{x=0}^{a'} {}^nC_x p^x q^{n-x} - S_0 v^n \sum_{x=0}^{a'} {}^nC_x (pu')^x (qd')^{n-x} \\
&= kv^n \sum_{x=0}^{a'} {}^nC_x p^x q^{n-x} - S_0 (pu'+qd') v^n \sum_{x=0}^{a'} {}^nC_x \left(\frac{pu'}{pu'+qd'}\right)^x \left(\frac{qd'}{pu'+qd'}\right)^{n-x} \\
&= kv^n \sum_{x=0}^{a'} {}^nC_x p^x q^{n-x} - S_0 \sum_{x=0}^{a'} {}^nC_x (vpu')^x (vqd')^{n-x} \\
&= kv^n F_{B(n,p)}(a') - S_0 F_{B(n,vpu')}(a')
\end{aligned}$$

where $a' = \log\left(k / S_0 (d')^n\right) / \log(u'/d')$. The delta of the debased call option is

$$\begin{aligned}
h_{0,n} &= \sum_{x=a}^{n-1} {}^{n-1}C_x (vpu')^x (vqd')^{(n-1)-x} + v^{n-1} \frac{\left(S_0 (u')^{a'} (d')^{n-a'} - k\right)}{S_0 (u'-d')} {}^{n-1}C_{a'-1} p^{a'-1} q^{n-a'} \\
&= \bar{F}_{B(n-1,vpu')}(a') + v^{n-1} \frac{\left(S_0 (u')^{a'} (d')^{n-a'} - k\right)}{S_0 (u'-d')} {}^{n-1}C_{a'-1} p^{a'-1} q^{n-a'}
\end{aligned}$$

rather than $\bar{F}_{B(n-1,vpu')}(a')$ because the exercise price *k* does not remain on the lattice after the sample space is changed as a result of the asset being demeaned with slack.

The delta of the debased put option is

$$\begin{aligned}
h_{0,n} &= -\sum_{j=0}^{a'-1} {}^{n-1}C_j (vpu')^j (vqd')^{n-1-j} - v^{n-1} \frac{\left(k - S_0 (u')^{a'} (d')^{n-a'}\right)}{S_0 (u'-d')} {}^{n-1}C_{a'} p^{a'} q^{n-1-a'} \\
&= -F_{B(n-1,vpu')}(a'-1) - v^{n-1} \frac{\left(k - S_0 (u')^{a'} (d')^{n-a'}\right)}{S_0 (u'-d')} {}^{n-1}C_{a'} p^{a'} q^{n-1-a'}
\end{aligned}$$

the additional term over $-F_{B(n-1,vpu')}(a'-1)$ similarly because of the changed lattice and sample space. The debased premiums in terms of u' are slightly different from those of p' because the discrete lattice of the sample space of the asset is altered rather than an alteration in the allocation and distribution of probability over the sample space. Apart from this they are exactly equivalent in structural form in that there is no return on the asset for which the option is priced and all probabilities of an inthemoney exercise are accurate for an asset with no return (junk probabilities), and equivalent in every other way to that debased by varying *p* including the limit to continuous time by making the step length small and the number of steps [and rebalances] infinite. The only difference is that the debasement here affects the minimum number of upsteps to expire inthemoney *a* rather than the probability of each upstep to expire inthemoney *p*. So the formulae for the debased prices are exactly the same except that one has p' and q' rather than *p* and *q* and the other has u' and d' rather than *u* and *d* (and consequently a' rather than *a*). In both cases the moneyness probabilities are junk probabilities for the actual asset to deliver since they are inaccurate and biased, and the return factor is nullified $(pu'v+qd'v)^n = 1$ so that the form of the debased formulae are exactly equivalent in terms of the binomial CMF weighted spot price and discounted exercise price. The mathematical implications are also exactly parallel (although it is more reasonable to modify return factors to demean the market asset) and are exactly equivalent for pseudo options when it is

possible to fully immunize the replicating portfolio from market dependent cashflow since the formulae, namely deltas and premiums, then do not depend on transition probabilities and binomial CMFs. The various methods of debasing the option premium are explored again in continuous time whereby the objective is to remean the debased premiums and reverse the debasement for accurate prices [for use in markets] rather than debase them, as here, to zero all net cashflows and void derivatives of market return and the benefits of the spot markets.

Note that the accurate premium, that not debased with slack, is obtainable exactly from the premium which is debased by taking slack from both the upstep and downstep return factors, $u' = u / \Delta$ and $d' = d / \Delta$ where $\Delta = (puv + qdv)$, which corresponds one-for-one with debasing a Brownian motion in continuous time [placing a sink directly on the market asset to demean it of market return $Y_n = S_n / (puv + qdv)^n$]. To obtain the accurate premium from the debased premium, in this case [for the premium priced for $Y_n = S_n / (puv + qdv)^n$], the spot price S_0 is remeaned for a future value by multiplying it by the total compounded slack for the n-step forward transition to option maturity $\Delta^n = (puv + qdv)^n$, which removes the total compounded slack and remeans the backward variable [a vertical shift upwards of the logarithm of the path of the demeaned asset]. That is, S_0 is replaced with $S_0 \Delta^n = S_0 (puv + qdv)^n$ wherever it occurs in the premium. This means **(1)** the additional factor of $(puv + qdv)^n$ in the coefficient for S_0 plus **(2)** the additional factor $(puv + qdv)^n$ in the standardized exercise price a' within the cumulative binomial probabilities (the complementary CMFs for the call option and the CMFs for the put option). This then **(1)** gives the spot value a future value for the deferral period of the derivative and **(2)** makes the expiry probabilities accurate for S. This corresponds one-for-one with that of continuous time by replacing S_0 with $S_0 \exp\left(\Delta(T - t_0)\right) = S_0 \exp\left((\mu + \frac{1}{2}\sigma^2 - r)(T - t_0)\right)$ to remean to the accurate premium by removing the slack from the debased premium [hence making the premium accurate for S rather than Y]. In markets, this is done so by reference to **market put-call parity**.

Solving the recurrence for the option PREMIUM debased by varying the parameter *u*

The recursion for the debased premium is the recursion for the accurate premium but instead with the debased return factors u' and d' built into the recursion and the boundary condition for the debased option accurate for the oxymoron risky bank account, which is conditional on the spot price $Y_0 = S_0$,

$f_0 = v^n \max\left(Y_n - k, 0 \mid S_0\right) = v^n \max\left(S_0 (u')^X (d')^{n-X} - k, 0 \mid S_0\right)$ with $X \sim B(n, p)$ for a call option and

$f_0 = v^n \max\left(Y_n - k, 0 \mid S_0\right) = v^n \max\left(S_0 (u')^X (d')^{n-X} - k, 0 \mid S_0\right)$ with $X \sim B(n, p)$ for a put option

The resulting premiums are the same as those for debasing by modifying the parameter for probability, apart from some small numerical difference due to the variation of the sample space for the modified return factors. This makes clear that it is not some alternative probability world that does not exist for which there is no risk, only a debased premium. From the backwards recursion of the option premium debased by taking slack from u

$f_0 = vpf_{u'} + vqf_{d'}$

and in one steps time this is either

$f_{u'} = vpf_{u'u'} + vqf_{d'u'}$ if the notional debased spot increases to $Y_{u'} = S_0 u'$ with frequency p

$f_{d'} = vpf_{u'd'} + vqf_{d'd'}$ if the notional debased spot decreases to $Y_{d'} = S_0 d'$ with frequency q

and again

$f_{u'u'} = vpf_{u'u'u'} + vqf_{u'u'd'}$ if the notional debased spot moves to $Y_{u'u'} = S_0 u'u'$ with frequency p^2

$f_{u'd'} = vpf_{u'd'u'} + vqf_{u'd'd'}$ if the notional debased spot moves to $Y_{u'd'} = S_0 u'd'$ with frequency $2pq$

$f_{d'd'} = vpf_{d'd'u'} + vqf_{d'd'd'}$ if the notional debased spot moves to $Y_{d'd'} = S_0 d'd'$ with frequency q^2

Unfolding the recursion for the debased premium f_0 to the forward variables at option expiration at the n^{th} step

$$
\begin{aligned}
f_0 &= vpf_{u'} + vqf_{d'} \\
&= vp\left(vpf_{u'u'} + vqf_{u'd'}\right) + vq\left(vpf_{u'd'} + vqf_{d'd'}\right) \\
&= v^2p^2 f_{u'u'} + 2v^2pqf_{u'd'} + v^2q^2 f_{d'd'} \\
&= v^2p^2\left(pf_{u'u'u'} + qf_{u'u'd'}\right) + 2v^2pq\left(pf_{u'd'u'} + qf_{u'd'd'}\right) + v^2q^2\left(pf_{d'd'u'} + qf_{d'd'd'}\right) \\
&= v^3p^3 f_{u'u'u'} + 3v^3p^2qf_{u'u'd'} + 3v^3pq^2 f_{u'd'd'} + v^3q^3 f_{d'd'd'} \\
&\ldots \\
&= v^n\sum_{x=0}^{n} {}^nC_x p^x q^{n-x} f_{(u')^x(d')^{n-x}}
\end{aligned}
$$

The boundary condition for the debased call option is $\max\left(Y_n - k, 0 \mid S_0\right) = \max\left(S_0(u')^x(d')^{n-x} - k, 0 \mid S_0\right)$ for $X \sim B(n, p)$ so that the option premium at the boundary for *x* upsteps is $f_{(u')^x(d')^{n-x}} = \max\left(S_0(u')^x(d')^{n-x} - k, 0\right)$ and the unfolded recursion solves for

$$
\begin{aligned}
f_0 &= v^n\sum_{x=0}^{n} {}^nC_x p^x q^{n-x} f_{(u')^x(d')^{n-x}} \\
&= v^n\sum_{x=0}^{n} {}^nC_x p^x q^{n-x} \max\left(S_0(u')^x(d')^{n-x} - k, 0\right) \\
&= \bar{F}_{B(n,p')}(a') - kv^n\bar{F}_{B(n,p)}(a')
\end{aligned}
$$

where $(pu'v + qd'v)^n = 1$. Similarly, the boundary condition for the debased put option is $\max\left(k - Y_n, 0 \mid S_0\right) = \max\left(k - S_0(u')^x(d')^{n-x}, 0 \mid S_0\right)$ for $X \sim B(n, p)$ so that the option premium at the boundary for *x* upsteps is $f_{(u')^x(d')^{n-x}} = \max\left(k - S_0(u')^x(d')^{n-x}, 0\right)$ and the unfolded recursion solves for

$$
\begin{aligned}
f_0 &= v^n\sum_{x=0}^{n} {}^nC_x p^x q^{n-x} f_{(u')^x(d')^{n-x}} \\
&= v^n\sum_{x=0}^{n} {}^nC_x p^x q^{n-x} \max\left(k - S_0(u')^x(d')^{n-x}, 0\right) \\
&= kv^n F_{B(n,p)}(a') - F_{B(n,p')}(a')
\end{aligned}
$$

where again $(pu'v + qd'v)^n = 1$ as a result of the asset and premium being debased.

Solving the recurrence for the DELTA of the option debased by varying the parameter *u*

The backwards recursion for the delta of the option debased by taking slack from the return factors is

$$h_0 = vpu'h_{u'} + vqd'h_{d'}$$

whereby the debased parameters feature in the recursion both in the coefficients and in the subscripts. The recursion of the delta of the debased option is solved by reference to changes in the demeaned asset since the recursion depends on the debased parameters that feature in the lattice of the boundary condition rather than the parameterization of the step generator. From $h_0 = vpu'h_{u'} + vqd'h_{d'}$, and conditional on the spot price $Y_0 = S_0$, in one steps time for the notional asset that has been debased of return this is either

$h_u = vpu'h_{u'u'} + vqd'h_{u'd'}$	if the notional debased spot increases to $Y_{u'} = S_0u'$ with frequency *p*
$h_d = vpu'h_{d'u'} + vqd'h_{d'd'}$	if the notional debased spot decreases to $Y_{d'} = S_0d'$ with frequency *q*

Similarly, in two steps time the recursion becomes

$h_{uu} = vpu'h_{u'u'u'} + vqd'h_{u'u'd'}$	if the notional debased spot moves to $Y_{u'u'} = S_0u'u'$ with frequency p^2
$h_{ud} = vpu'h_{u'd'u'} + vqd'h_{u'd'd'}$	if the notional debased spot moves to $Y_{u'd'} = S_0u'd'$ with frequency $2pq$
$h_{dd} = vpu'h_{d'd'u'} + vqd'h_{d'd'd'}$	if the notional debased spot moves to $Y_{d'd'} = S_0d'd'$ with frequency q^2

and so forth. Unfolding the recurrence over the backward variables of *Y* to the forward variables at option expiration

$$
\begin{aligned}
h_0 &= vpu'h_{u'} + vqd'h_{d'} \\
&= vpu'\left(vpu'h_{u'u'} + vqd'h_{u'd'}\right) + vqd'\left(vpu'h_{u'd'} + vqd'h_{d'd'}\right) \\
&= v^2(pu')^2 h_{u'u'} + 2v^2(pu')(qd')h_{u'd'} + v^2(qd')^2 h_{d'd'} \\
&= v^2(pu')^2\left(pu'h_{u'u'u'} + qd'h_{u'u'd'}\right) + 2v^2(pu')(qd')\left(pu'h_{u'd'u'} + qd'h_{u'd'd'}\right) + v^2(qd')^2\left(pu'h_{d'd'u'} + qd'h_{d'd'd'}\right) \\
&= v^3(pu')^3 h_{u'u'u'} + 3v^3(pu')^2(qd')h_{u'u'd'} + 3v^3(pu')(qd')^2 h_{u'd'd'} + v^3(qd')^3 h_{d'd'd'} \\
&\ldots \\
&= \sum_{x=0}^{n-1} {}^{n-1}C_x (vpu')^x (vqd')^{n-1-x}\, h_{(u')^x (d')^{n-1-x}}
\end{aligned}
$$

At the boundary at the forward variables, the call option delta is one for that part of the range where the option is inthemoney and zero for that part where the option is outofthemoney on the forward distribution of the demeaned asset. If the exercise price is on the lattice of the market asset, then it is not on the lattice of the demeaned asset since debasing the return factors with slack changes the sample space. Therefore

$$
h_{(u')^x (d')^{n-1-x}} = \begin{cases} 1 & for\ x > a'-1 \\ \dfrac{\left(S_0(u')^{a'}(d')^{n-a'} - k\right)}{S_0(u')^{a'-1}(d')^{n-a'}(u'-d')} & for\ x = a'-1 \\ 0 & for\ x < a'-1 \end{cases}
$$

where $a' = \log\left(k / S_0(d')^n\right)/\log(u'/d')$ and $x = a'-1$ is the node just before expiry whereby a step upwards moves the option into the money for the long trade to deliver and a step downwards moves the option outofthemoney [for a turned down long trade] with respect to the demeaned asset for which the option is priced. So the call option delta is then

$$
\begin{aligned}
h_0 &= \sum_{x=a'}^{n-1} {}^{n-1}C_x (vpu')^x (vqd')^{n-1-x} \times 1 + \sum_{x=a'-1}^{a'-1} {}^{n-1}C_x (vpu')^x (vqd')^{n-1-x} \frac{\left(S_0(u')^{a'}(d')^{n-a'} - k\right)}{S_0(u')^{a'-1}(d')^{n-a'}(u'-d')} \\
&= \bar{F}_{B(n-1,vpu')}(a') + \frac{\left(S_0(u')^{a'}(d')^{n-a'} - k\right)}{S_0(u'-d')} {}^{n-1}C_{a'-1}(vp)^{a'-1}(vq)^{n-a'}
\end{aligned}
$$

The put option delta is zero for that part of the range where the option is outofthemoney and instead, negative one [a protected position for the short trade] for that part where the option is inthemoney on the forward distribution of the demeaned asset. Similarly, as the call option, if the exercise price is on the lattice of the market asset then it is not on the lattice of the demeaned asset because debasing the return factors with slack changes the sample space. So the delta at the boundary is

$$
h_{u^x d^{n-1-x}} = \begin{cases} 0 & for\ x > a' \\ -\dfrac{\left(k - S_0(u')^{a'}(d')^{n-a'}\right)}{S_0(u')^{a'-1}(d')^{n-a'}(u'-d')} & for\ x = a' \\ -1 & for\ x < a' \end{cases}
$$

where $a' = \log\left(k / S_0(d')^n\right)/\log(u'/d')$ and $x = a'$ is the node just before expiry whereby a step upwards moves the option out of the money [for a turned down short transaction] and a step downwards moves the option into the money for the short trade to deliver with respect to the demeaned asset for which the option is priced. So the option delta is then

$$
\begin{aligned}
h_0 &= -\sum_{x=0}^{a'-1} {}^{n-1}C_x (vpu')^x (vqd')^{n-1-x} \times 1 - \sum_{x=a'}^{a'} {}^{n-1}C_x (vpu')^x (vqd')^{n-1-x} \frac{\left(k - S_0(u')^{a'}(d')^{n-a'}\right)}{S_0(u')^{a'}(d')^{n-a'-1}(u'-d')} \\
&= -F_{B(n-1,vpu')}(a'-1) - \frac{\left(k - S_0(u')^{a'}(d')^{n-a'}\right)}{S_0(u'-d')} {}^{n-1}C_{a'}(vp)^{a'}(vq)^{n-a'-1}
\end{aligned}
$$

Solving the recurrences for the debased option premium and delta which is debased by taking slack from the return parameters u and d applies to any method which demeans the underlying asset with u' and d' so that it is void of market return. Such other techniques are discussed now.

(3) Other methods of demeaning the market asset *S*

There are many methods of killing the return of the market asset for a debased portfolio and an equivalent demeaned asset Y that is zero returning net of the cash rate, all of which zero the net cashflows [of the replication that nullifies the option], each by varying one set of parameters (p and q or u and d) of the walk to null the excess return by setting $puv + qdv$ to one and solving

$$puv + qdv = 1$$

This is what return-neutered valuation tries to say is *an alternative Q world that does not exist for derivative pricing*. All it is, rather, is a portfolio that is debased by **debasing the market asset and premium with a slack variable that kills all net cashflow [however this occurs], making the portfolio void of the benefits of trading a market asset.** There is nothing that "does not exist" in a debased portfolio, nor is it another alternative world. It is nothing other than a debased portfolio that trades a demeaned and redundant asset Y to replicate, offset, and flatline a mispriced option priced accurately only for this demeaned and redundant asset. Other methods for debasing an option portfolio by demeaning the market asset are considered now.

With consideration to the constraint $u = 1 / d$ in addition to $p = 1 - q$, techniques for debasing by a proportional factor Δ that increases the magnitude of upsteps and decreases the magnitude of downsteps are discussed now; this is either **(i)** $p/\Delta + q\Delta = 1 + r$ or **(ii)** $pu/\Delta + qd\Delta = 1 + r$.

(i) The one method of demeaning the underlying asset with a modification of the proportional growth factor u that conforms to the additional restriction on the parameterization that the proportional upstep be reciprocal to that of the proportional downstep $u = 1 / d$ (which keeps the sample space of the proportional walk symmetric on logarithms) is that by the parameter letting u vary for the slack between return and no return by solving the equation resulting from setting the mean return to the cash rate $pu' + pd' - 1 = r$ subject to $\Delta = u' = 1 / d'$

$$pu' + \frac{q}{u'} = 1 + r \qquad \Rightarrow \qquad 0 = p(u')^2 - v^{-1}u' + q$$

which results in the debased return factors

$$u' = (1+r)\frac{1+\sqrt{1-4pqv^2}}{2p} \qquad \text{and} \qquad d' = \frac{1}{u'} = (1+r)\frac{1-\sqrt{1-4pqv^2}}{2q}$$

This dispenses with the original values of u and d altogether as a result of keeping the two reciprocal similarly as p' and q' dispenses with p and q altogether as a result of the two being complementary probabilities.

(ii) If there is not the restriction that the ratios be reciprocal, then the return parameters can alternatively and more flexibly debased with the proportional slack variable Δ modulating the return factors ($\Delta > 1$) by taking the slack from u and adding it back to d on logs, making the upstep returns smaller ($u' < u$) and downstep losses also smaller ($d' > d$). This is not the most effective method of debasing an asset but it is consistent with a proportional moderator which moderates both the mean and variance [killing the market return in doing so]. Solving $pu' + qd' = 1 + r$ in terms of $u' = u/\Delta$ and $d' = d\Delta$ for Δ

$$p\frac{u}{\Delta} + qd\Delta = 1 + r \qquad \Rightarrow \qquad 0 = qd\Delta^2 - v^{-1}\Delta + pu$$

which solves for

$$\Delta = (1+r)\frac{1-\sqrt{1-4v^2pqud}}{2qd}$$

This is defined so long as $4v^2pqud < 1$ and is exactly the same as the previous u' and d' with the added flexibility that u and d need not be proportional. Since

$$\frac{1}{\Delta} = v\frac{2qd}{1-\sqrt{1-4v^2pqud}} = v\frac{2qd}{1-\sqrt{1-4v^2pqud}}\frac{1+\sqrt{1-4v^2pqud}}{1+\sqrt{1-4v^2pqud}} = (1+r)\frac{1+\sqrt{1-4v^2pqud}}{2pu}$$

the factors debased with slack $u' = u/\Delta$ and $d' = d\Delta$ are then

$$u' = (1+r)\frac{1+\sqrt{1-4v^2pqud}}{2p} \qquad \text{and} \qquad d' = (1+r)\frac{1-\sqrt{1-4v^2pqud}}{2q}$$

Whenever *u* and *d* are proportional ($ud = 1$), this is exactly that of the previous formula **(i)**. Both of these methods **debase the asset and premium and zero all net cashflows.**

The problem with both of the two methods is that the walk variance is changed in demeaning the asset as there is only one degree of freedom for either probability parameters or return parameters to debase the asset (subject to $p = 1 - q$ and $u = 1/d$) which affect both the two statistical attributes of the walk, the mean and variance $pu + qd$ and $pq(u-d)^2$. This does not happen, for logarithms, when the slack variable reduces both upsteps and downsteps **impartially as a sink** on the walk variable and as the technique of focus for *u'* and *d'*.

Modifying return factors after freeing up the constraint of proportional return factors allows two parameters to be modified to match with the two parameters of the asset, the mean and variance, and means that the debasing factor Δ need not be proportional.

(iii) From the debased parameters for probability $p' = (1+r-d)/(u-d)$ and $p' = (u-1-r)/(u-d)$, if these are used to debase via the return parameters for the walk dynamics, then similarly as the earlier heuristic argument $E(Y_1 \mid Y) = p(up'/p)Y_0 + q(dq'/q)Y_0 = pu'Y_0 + qd'Y_0 = (1+r)Y_0$ for a debased asset with

$$u' = \frac{u(1+r-d)}{p(u-d)} \qquad \text{and} \qquad d' = \frac{d(u-1-r)}{q(u-d)}$$

which debases the asset and premium and zeroes all net cashflows. Here, the debased parameters do not feature as coefficients to the exercise price but rather feature in this term to the premium as a moderator in the nominalized exercise price *a*, and bias the probabilities directly through *a*. Note that the upstep factor is then reduced below 1 ($u' < 1$) and the upstep factor is increased above 1 ($d' > 1$) which is a little silly but still results in everything being debased, and a debased premium on par with any of the other methods including that with $p' = p - \Delta$. This is the least sensible method and is here only as a heuristic.

(iv) Decreasing return or visa-versa should not increase the variance and ideally, modifying the return profile should leave the walk variance intact. Without taking logarithms, the only method to do so for a proportional walk is by subtracting the excess market return directly from both the return factors. For this one alternative that leaves the step variance unchanged, that whereby a constant is subtracted from both *u* and *d*, letting Δ be the additive slack that debases the return factors by reducing both the upstep and downstep factors for $u' = u - \Delta$ and $d' = d - \Delta$

$$\begin{aligned} &p(u-\Delta) + q(d-\Delta) = 1+r \\ &pu + qd - (p+q)\Delta = 1+r \\ &\Delta = pu + qd - 1 - r \end{aligned}$$

$$\begin{aligned} u' &= u - \Delta = u - pu - qd + 1 + r = 1 + r + q(u-d) \\ d' &= d - \Delta = d - pu - qd + 1 + r = 1 + r - p(u-d) \end{aligned}$$

which leaves the walk variance unchanged and zeroes all net cashflows by reducing all movements by a flat amount, i.e., **debases the asset and premium and zeroes all net cashflows**.

(v) Out of interest, letting Δ be the component of return that demeans the asset by putting $u' = 1 - \Delta$ and $d' = 1 - \Delta$ dispensing with *u* and *d* entirely

$$\begin{aligned} &p(1-\Delta) + q(1-\Delta) = 1+r \\ &1 - \Delta = 1 + r \\ &\Delta = -r \end{aligned}$$

which means that $u' = 1 + r$ and $d' = 1 + r$ so the debased variable also has no variance and is therefore a **risk free bank account proper** rather than an **oxymoron risky bank account** which earns the cash rate only but has full market risk or even more. The option premium in this case limits to a pseudo option on the discount factor, $S_0 - kv^n$ if the spot is inthemoney ($S_0 > k$) and 0 otherwise, **and consequently zeroes all net cashflows**.

(vi) Freed from the constraint $u = 1/d$, alternatively making both *u* smaller and *d* larger with additive Δ for $u' = u - \Delta$ and $d' = d + \Delta$

$$p(u-\Delta)+q(d+\Delta)=1+r$$
$$pu+qd-(p-q)\Delta=1+r$$
$$\Delta=(pu+qd-1-r)/(p-q)$$

$$u'=u-(pu+qd-1-r)/(p-q)=\big(1+r-q(u+d)\big)/(p-q)$$
$$d'=d+(pu+qd-1-r)/(p-q)=\big(p(u+d)-1-r\big)/(p-q)$$

which debases the asset and premium and zeroes all net cashflows.

(vii) As a simplification, letting Δ be the component of return that demeans the asset by putting $u'=1+\Delta$ and $d'=1-\Delta$ again dispensing with *u* and *d* entirely

$$p(1+\Delta)+q(1-\Delta)=1+r$$
$$1+(p-q)\Delta=1+r$$
$$\Delta=r/(p-q)$$

$$pu+qd=1+r$$
$$pq(u-d)^2=pq\big(2r/(p-q)\big)^2=4pqr^2/(p-q)^2$$

for $u',d'=1\pm r/(p-q)$ **which debases the asset and premium and zeroes all net cashflows.**

You may experiment at will; these methods are good for playing around with, and good for experimenting with[17], the methods to debase an asset and zero the net cashflows of an option replicating portfolio by debasing the cashflow of market return. It is the purpose of this book, though, only to make clear that the debasement of options and other derivatives is not at all sensible, and **so is presenting an argument against the debasement of derivatives** in favour of derivative markets for which **(1)** risk and return trades with accurate prices and **(2)** that fully recognizes both return and the economic value of derivatives.

The Total Derivative of the Option on the Demeaned Asset Y

Consideration is now given to the portfolio whereby the asset demeaned after the portfolio has been debased is kept void of market return and notionally traded to replicated the option at the debased price. This is a discussion **in prelude to** remeaning the asset [for that which is traded] whilst keeping the premium and deltas debased for which it is hoped that the market return will source the mispricing in the debased premium [earned from flatlining the option] and keep the portfolio return flat and debased at *r* and free from market return. This is considered shortly. This topic therefore considers the case where all variables remain debased, rather than just the premium and deltas; for portfolio returns and then for the NPV of cashflows of the replicating portfolio that trades the demeaned asset *Y* to replicate the option at the debased price which is accurate for *Y* and *Y* only. In both cases, the portfolios remain debased at the debased price for the premium. For that considered here, the premium is accurate for the notionally traded demeaned asset *Y* whilst in the case whereby the asset is remeaned, the debased premium is inaccurate for the traded asset *S* being used to replicate the option all the way up to option maturity. In both cases, the debased premium is priced for an asset that has been demeaned of all market return.

The demeaned asset with the fake debased probability parameter $p' = (v^{-1} - d) / (u - d)$, and that for which the debased option premium is then priced, is binomial with binomial index *n* for the number of steps to option maturity as for the accurate premium and the forward transition of the market asset *S*, but instead with the probability of an upstep debased with slack p'. It was shown that substituting the debased probability p' into the coefficient of the delta in the total derivative of the option premium, which is a coefficient of the mean rate of effective market return $pu + pd - 1$ per binomial step, results in a growth rate fixed flat at the cash rate which exactly offsets the discount factor and the time value of cash, i.e., is null net of cash

$$p'u+q'd-1=u\frac{v^{-1}-d}{u-d}+d\frac{u-v^{-1}}{u-d}-1=\frac{uv^{-1}-ud+ud-dv^{-1}}{u-d}-1=v^{-1}-1=r$$

and is synonymous with the result of that by definition of the demeaned asset *Y*. The total derivative of the option premium with the debased probabilities becomes

$$\frac{\Delta f_{j,n}}{\Delta j}=rf_{j,n}-(p'u+q'd-1)Y_j h_{j,n}+h_{j,n}\frac{\Delta Y_j}{\Delta j}$$
$$=rf_{j,n}-rY_j h_{j,n}+h_{j,n}\frac{\Delta Y_j}{\Delta j}$$

[17] by this it is meant debasing the portfolio by varying the parameters (*p* and *q* or *u* and *d*) with the various forms of slack variable on a spreadsheet of the cashflows for the portfolio which replicates the accurate premium.

where Y_j is the variable of the demeaned asset. The coefficients of the terms in the total change in the debased option premium, the term in *f* and that in delta and the demeaned asset *Yh*, are the same at *r* and whence factor for a flat portfolio return net of the replicating effects $h_{j,n}\Delta Y_j$ and the mandatory rebalance in delta $Y_{j+1}\Delta h_{j,n}$ which is sufficient for, and only for, a walk with a binomial basis function (i.e., a walk with two-valued steps). It is not possible to debase an option portfolio for a flat portfolio return for any other variable without the possibility of the delta updates eating into the portfolio return leaving it randomly variable rather than flat at *r*.

The Return Components of the Debased Option Replicating Portfolio that Replicates with the Demeaned Asset Y

In the case of the debased portfolio with probability varied for slack for the return-neutered methodology (or any other method that debases an option portfolio of market return), whereby the market asset upon which the option is priced is demeaned to the cash rate yet remains fully inclusive of market risk, the return on the underlying asset **simply offsets the effects of the time value of money which, in many circumstances, is simply the rate of inflation (or less)**. The value changes in the contents of the portfolio, which is the between node returns and the proportional change occurring between the nodes of the lattice, are fixed at the cash rate and are hence flat for each step of the lattice between the buys and sells for the delta updates

$$\begin{aligned}(h_{j,n}\Delta Y_j - \Delta f_{j,n}) &= r\,h_{j,n}Y_j - r\,f_{j,n}\\ &= r\,(h_{j,n}Y_j - f_{j,n})\\ &= r\,\Pi_{j,n}\end{aligned}$$

that is, $\Pi_{j,n} = (pu+qd)Y_j h_{j,n} - (1+r)\,f_{j,n} + Y_{j+1}\Delta h_{j,n}$ with either p' or u', becomes

$$\begin{aligned}\Pi_{j+1,n} &= (1+r)Y_j h_{j,n} - (1+r)\,f_{j,n} + Y_{j+1}\Delta h_{j,n}\\ &= (1+r)\Pi_{j,n} + Y_{j+1}\Delta h_{j,n}\end{aligned}$$

For a single step this is

$$\begin{aligned}\Pi_{1,n} &= (1+r)Y_0 h_{0,n} - (1+r)\,f_{0,n} + Y_1\Delta h_{0,n}\\ &= (1+r)\Pi_{0,n} + Y_1\Delta h_{0,n}\end{aligned}$$

so that, net of the outstanding rebalance due to complete the total change of the portfolio, the portfolio is debased with the constraint held at $\Pi_0 = v\Pi_u = v\Pi_d$ surreptitiously demeaning the return on the market asset (by letting slack vary with *p* or *u*) and holding the between node changes fixed flat at the cash rate rather than an interaction between returns on the market asset and the cash rate on the option, the latter of which simply being time decay in the discount factor. The proof of this parallels that for the unbiased option; substituting in the total derivative of the option premium into the total derivative of the option replicating portfolio, in the particular case whereby the underlying asset grows at a rate equivalent to that of the cash rate, results in the flat portfolio return at the cash rate r. The delta-neutral portfolio does not earn a flat rate at cash otherwise, i.e., whenever either the market asset does not earn the cash rate $pu + qd - 1 \neq r$ or the portfolio and premium have not been debased. In the case where it does not, as proved, the between node portfolio return is not flat and instead differs from period-to-period, dependent on the path of the spot value of the underlying asset and the path of the buys and sells in the replicating delta, as the return on the asset holding interacts with the rate at which values are accumulated at the cash rate, i.e., earning $pu + qd - 1$ on the asset and incurring r on the option premium [which continually vary in relative composition within Π]. In this case, when the return on the market asset exceeds the cash rate, market return is earned over each possible path in proportion to the path of delta. Each of the proofs for the debased portfolio are exactly the same as the statistically unbiased and accurate premium for the market asset, by simply demeaning the market asset by putting $pu+qd=1+r$ to demean it with a debased probability or growth factor by solving $pu+qd=1+r$ by varying *p* or *u* with slack. In continuous time, the equivalent debased portfolio occurs by the same constraint $d\Pi_t = r\Pi_t dt$ which surreptitiously forces $\mu+\frac{1}{2}\sigma^2 = r$ by varying the parameter μ with slack Δ, i.e., putting $\mu' = \mu - \Delta$ so that $\mu'+\frac{1}{2}\sigma^2 = r$ and $(\mu-\Delta)+\frac{1}{2}\sigma^2 = r$ for which solving for Δ results in the slack variable $\Delta = \mu+\frac{1}{2}\sigma^2 - r$ which demeans the market Brownian motion to *Y* and debases the portfolio for a debased premium. The same slack then reappears with volatility skew when the debased price is set to market prices, so that the slack introduced via the mean μ with $d\Pi_t = r\Pi_t dt$ is then backed out again via the volatility σ causing the skew in σ vs *k* since the bias of the debased price is a function of *k* [as

well as term to maturity]. The slack variable introduced via the mean $\Delta = \mu + \frac{1}{2}\sigma^2 - r$ is then numerical when backed out via volatility. The backed out numerical skewed volatility is then $\sigma^{IV} = c\sigma$ where $c = c(\Delta,k)$ is proportional slack and a function of Δ introduced via the mean.

Each of the proofs for the portfolio return components are obtainable directly from that of the premium accurate for the market asset *S* by demeaning the return to the cash rate exactly as for the formulae for the premiums.

Proof by discrete calculus

The option premium $f_{j,n}$ for a fixed option maturity at the forward time variable *n* is a function of two variables; the backward time variable *j* and the backward walk variable of the demeaned asset Y_j. Applying the divided difference operator to $\Pi_j = h_{j,n}Y_j - f_{j,n}$ for the between node changes with respect to the time index

$$\frac{\Delta\Pi_j}{\Delta j} = Y_{j+1}\frac{\Delta h_{j,n}}{\Delta j} + h_{j,n}\frac{\Delta Y_j}{\Delta j} - \frac{\Delta f_{j,n}}{\Delta j}$$

and substituting in the total derivative of the debased option premium

$$\begin{aligned}\frac{\Delta\Pi_{j,n}}{\Delta j} &= Y_{j+1}\frac{\Delta h_{j,n}}{\Delta j} + h_{j,n}\frac{\Delta Y_j}{\Delta j} - \left(rf_{j,n} - rY_j h_{j,n} + h_{j,n}\frac{\Delta Y_j}{\Delta j}\right)\\ &= Y_{j+1}\frac{\Delta h_{j,n}}{\Delta j} + rY_j h_{j,n} - rf_{j,n}\\ &= Y_{j+1}\frac{\Delta h_{j,n}}{\Delta j} + r\Pi_{j,n}\end{aligned}$$

after canceling the terms in the offsetting replicating delta of the demeaned asset $h_{j,n}\Delta Y_j$. The total change in the debased portfolio is **(1)** the cash rate on the debased portfolio plus **(2)** the delta update in *Y*. The total change in the debased portfolio is stated as $\Delta\Pi_{j,n} = r\Pi_{j,n}$ net of the delta update due $Y_{j+1}\Delta h_{j,n}$. This condition $\Delta\Pi_{j,n} = r\Pi_{j,n}$ characterizes debased option portfolios, those that trade and replicating with the demeaned asset as well as those that trade and replicate with the market asset to replicate the debased premium. This accounts for all methods, bar none, of debasing a portfolio by demeaning the market return resulting in zero net cashflows.

The proof is complete.

Inclusive of the mandatory delta update $Y_{j+1}\Delta h_j$ at the node if the demeaned asset was being traded, the total difference change in the portfolio is then

$$\Delta\Pi_j = r\Pi_j + Y_{j+1}\Delta h_j$$

which is a first order linear difference equation with an exogenous input of the delta updates at the end of each step to maintain the offsetting delta. The proof by combinatorial relations for $\Delta\Pi_j = r\Pi_j + Y_{j+1}\Delta h_j$ is simply a repeat of the mathematics with $puv + qdv = 1$ (by debasing with variations in either *p* or *u* for the slack) and for this reason is omitted as it is simply that for the accurate premium and market asset *S* by putting $p = p'$ or $u = u'$ so that $puv + qdv = 1$ for a portfolio that trades *Y* in order to replicate a premium accurate for *Y*. The delta update is in/with the demeaned asset, as this is the asset that is being used to notionally replicate the debased premium for the option priced for this asset within the debased portfolio. The portfolio return and cashflows for a debased portfolio and premium that trades the market asset *S* is considered after the NPV of cashflows which notionally trade *Y*. This is what risk-neutral valuation has misconstrued as a "return to the real world", whereby the asset is remeaned by removing the slack in p' (or u') so that the probabilities or return factors are returned back to the proper parameters whilst keeping the debased premium with the hope of earning back the mispricing in the debased price with the trading returns of the remeaned market asset earned by replicating and flatlining the option, i.e., the returns earned by flatlining the option are used to subsidize the mispricing in the debased premium to keep the portfolio return flat at *r* and cashflows void of return. If an option is not flatlined with a replicating asset, then there is no way of subsidizing the mispricing in the debased premium and the debased price is outright wrong. The portfolio that is debased and trades the notionally demeaned asset is being considered here in prelude and as an intermediate step to the case of subsidizing a debased premium with returns earned on trading *S* to flatline the option.

The NPV of Cashflows for the Debased Option Replicating Portfolio that Replicates with the Demeaned Asset Y

From the relation of the differential for the lattice changes between the nodes in the portfolio for an option priced for the demeaned asset *Y*, rather than the market asset *S*, consideration is now given to the portfolio Π for which the debased option premium $f_{j,n} = v^n E\left(\max(Y_n - k, 0) \mid Y_j\right)$ is replicated and offset by trading the demeaned asset *Y* for which the debased premium is accurate. Inclusive of the delta update by hypothetically trading the demeaned asset *Y*, the total change in the portfolio is then $r\Pi_j$ plus the update in *Y*, $Y_{j+1}\Delta h_j$

$$\Delta\Pi_j = r\Pi_j + Y_{j+1}\Delta h_j$$

The total differential of the replicating portfolio $\Delta\Pi_j = r\Pi_j + Y_{j+1}\Delta h_j$ is here treated as a first order linear difference equation which is summed into the cashflows, as for that with the accurate premium. That which replicates the option at the debased price by trading the actual market asset, $\Delta\Pi_j = r\Pi_j + S_{j+1}\Delta h_j$, is considered in the next section. The portfolio value and rebalance are the dependent variables and *j* is the independent variable similarly to that for the accurate premium and market asset. Moving the terms for the portfolio to one side of the equation and, to the other side, the term representing the delta update for the buys and sells of units of the demeaned asset to rebalance the portfolio

$$Y_{j+1}\Delta h_j = \Delta\Pi_j - r\Pi_j$$

The discrete time integrating factor is $v^{j+1} = (1+r)^{-(j+1)}$. Multiplying both sides of the equation by the integrating factor,

$$v^{j+1} Y_{j+1}\Delta h_j = v^{j+1}\Delta\Pi_j - r v^{j+1}\Pi_j$$

which puts all terms into present values. Factoring the right hand side in terms of the difference operator Δ

$$v^{j+1} Y_{j+1}\Delta h_j = \Delta\left(v^j \Pi_j\right)$$

and summing over the total replication period of the hedge portfolio from the time the option is written up to the forward time variables at time *n* at option expiry, over the backward time variable *j* (i.e., over the life of the option from inception and exchange of premium at time zero to maturity and closing out the portfolio at time *n*)

$$\begin{aligned}\sum_{j=0}^{n-1} v^{j+1} Y_{j+1}\Delta h_j &= \sum_{j=0}^{n-1} \Delta\left(v^j \Pi_j\right) \\ &= \left(v\Pi_1 - \Pi_0\right) + \left(v^2\Pi_2 - v\Pi_1\right) + \ldots \\ &\quad \ldots + \left(v^{n-1}\Pi_{n-1} - v^{n-2}\Pi_{n-2}\right) + \left(v^n \Pi_n - v^{n-1}\Pi_{n-1}\right) \\ &= v^n \Pi_n - \Pi_0\end{aligned}$$

since the sum is telescopic, the sum of a difference. Hence

$$v^n \Pi_n - \sum_{j=0}^{n-1} v^{j+1} Y_{j+1}\Delta h_j - \Pi_0 = 0$$

which is the equation of the NPV of cashflows for the demeaned asset *Y* and the premium priced for the demeaned asset, and consequently biased for the market asset to deliver. The NPV of the portfolio cashflows can alternatively be written in terms of the components to the portfolio with $\Pi_0 = h_0 Y_0 - f_0$ and $\Pi_n = h_n Y_n - f_n$

$$v^n (h_n Y_n - f_n) - \sum_{j=0}^{n-1} v^{j+1} Y_{j+1}\Delta h_j - (h_0 Y_0 - f_0) = 0$$

where $f_0 = v^n E\left(\max(Y_n - k, 0) \mid Y_0\right)$ and $f_n = \max(Y_n - k, 0)$ for a call and $f_0 = v^n E\left(\max(k - Y_n, 0) \mid Y_0\right)$ and $f_n = \max(k - Y_n, 0)$ for a put. Given that within this portfolio it is the demeaned asset that is traded and rebalanced, the equation of net cashflows is saying that the cost of **(i)** establishing the portfolio $h_0 Y_0 - f_0$, **(ii)** the present value of the delta rebalances, and **(iii)** the present value of the closing out proceeds $v^n \Pi_n = v^n (h_n Y_n - f_n)$ is zero over each path of the underlying walk of the demeaned asset *Y*, the resulting path of the option, and its replicating delta. Each path is zeroed simply as a result of zeroing the returns, which are an artifact of drift of a proportional diffusion, and flatlines entirely the payoff profile of the option. Although there is no variation in

outcome following closing out the hedge portfolio, this is because there is no return on all of the trading in the underlying asset and there is no option payoff profile at the option expiration, i.e., the replication has turned the option into **a redundant security void of a payoff profile**. If the option expires for the actual market asset to deliver, then there is the additional cost of delivering the market asset since the portfolio is [notionally] holding the demeaned asset and there is an inconsistency in the deltas and expiry value of the portfolio. The expected present value of this additional cost added to the cashflows is another expression for the statistical bias of the debased price and simply amounts to the difference between the two premiums, the debased price and the accurate price for *S*. This is the definition of the bias and this additional cost at the boundary results from remeaning all functions, rather than remeaning the demeaned asset *Y* to *S* whilst keeping the premiums and deltas debased which is considered now.

The Return Components of the Debased Option Replicating Portfolio that Replicates with the Market Asset S

The debased portfolio earns the cash rate *r* net of changes in the delta since the net effect of the appreciation in the position in the asset and the short option (the value changes) is set fixed and flat at the cash rate over each step, as a result of the constraint $\Pi_0 = v\Pi_d = v\Pi_u$ which turns gold (Au79) into lead (Pb82). Western civilization has an aversion to this. Since the asset actually earns the rate of market return $pu+qd-1$ rather than the cash rate *r* regardless of any option that is written on it or any constraint placed on the portfolio, when the market asset is traded to replicate the option within the debased portfolio, it is the option premium that then adjusts to the artificial return constraint and is how the bias and debasement is accounted for within the debased option portfolio, i.e., via a mispriced premium. The mean appears to vanish whereby the mean is debased from the asset and the premium by setting $\Pi_0 = v\Pi_d = v\Pi_u$ and then returned back by trading the market asset, noting importantly that it [the mean] is not returned back to the premium so the premium remains debased and mispriced for the market asset to deliver. This is essentially remeaning the asset [by removing the slack from p' or u'] whilst keeping the portfolio, premium, and deltas debased. So, net of the outstanding delta rebalance due, when the market asset is traded within the portfolio, the market asset still earns the effective market rate of return $pu+qd$ regardless, and the short option would ordinarily incur the cash rate $1+r$. With the constraint of a fixed portfolio return flat at the cash rate, the rate incurred on the premium is forced to sink all of the market return earned on *S*. Constraining the replicating portfolio flat at the cash rate $\Pi_1 = (1+r)\Pi_0$ and noting that, despite the debasing constraint, the delta position in the market asset *S* earns the market return $pu+qd$ anyway rather than the cash rate $1+r$

$$\begin{aligned}\Pi_1 &= (1+r)\Pi_0 \\ &= (1+r)S_0h_0 - (1+r)f_0 \\ &= (pu+qd)S_0h_0 - (1+r)f_0 - (pu+qd)S_0h_0 + (1+r)S_0h_0 \\ &= (pu+qd)S_0h_0 - \big((1+r)f_0 + (pu+qd-1-r)S_0h_0\big)\end{aligned}$$

showing that the market asset earns the market rate whilst the short option premium incurs a massive loss from having to sink the returns of the market asset to keep the portfolio return flat at *r* and gain ground in order to catch up with the accurate premium as expiry nears, i.e., the option incurs

$$\big((1+r)f_0 + (pu+qd-1-r)S_0h_0\big)/f_0 \qquad > pu+qd > 1+r$$

rather, and much larger than, $(1+r)f_0/f_0 = 1+r$. This heuristic is proved shortly by separating out the slack from the debased parameters. The short option incurs the cash rate on the premium plus a massive additional loss, in proportion to delta, at the market rate $(pu+qd-1-r)S_0h_0$ due to the mispricing. This kills any return that the market asset may earn. The net change is then

$$\begin{aligned}\Delta\Pi_j &= r\Pi_j \\ &= rS_jh_j - rf_j \\ &= (pu+qd-1)S_jh_j - \big(rf_j + (pu+qd-1-r)S_jh_j\big)\end{aligned}$$

The rate incurred on the option is well in excess of the both the cash rate and the market rate of return since the value of the asset in the portfolio, in proportion to delta, always exceeds the value of the option (for a call)

$$(1+r)f_0 + (pu+qd-1-r)S_0h_0 > (pu+qd)f_0$$

$$rf_0 + (pu + qd - 1 - r)S_0h_0 > (pu + qd - 1)f_0$$

$$\begin{aligned} r + (pu + qd - 1 - r)S_0h_0 / f_0 &> r \\ &> pu + qd - 1 \end{aligned} \qquad \text{since } S_0h_0 > f_0$$

This **additional loss amortizes the bias** with **subsidies from the market asset going directly to the mispricing in the debased premium**. The bigger the return on the market asset, the bigger the loss on the option arising from the mispricing; so whenever the traded asset earns any return, it is sunk by the mispricing in the premium. The mispricing kills the market return and the replication, to earn back the mispricing, kills the option by flatlining the payoff profile.

For these reasons, this thesis sometimes refers to this as **subsidized option valuation** whereby the asset returns earned by flatlining the option are used to misprice it. This **defies common sense [it is outright stupidity]**. Subsidizations of mispriced options that are all flatlined and made redundant in order to misprice them is **a pointless circularity**. If one wants the option, then one must not replicate and kill the option. If one does not replicate with an offsetting asset and kill the option, then there is no means by which to subsidize the mispricing so in which case (the majority of traded options), the option price must be the statistically accurate price. The only sensible market rationale for options is that which fully recognizes return, which is that of unbiased and statistically accurate premiums and full recognition of market return in derivatives, as proposed by this book of this thesis. Returns are the **essence of markets** and denying securities of return takes away all incentives to write and trade them, and a market of subsidized option valuation is better closed down.

(i) Portfolio debased by letting probability *p* vary for the slack between return and no return

Proof 1

For the portfolio and premium debased by varying the parameter p (subject to $q = 1 - q$), the recursion for the debased option premium is $f_0 = vp'f_u + vq'f_d$, where p' and q' are the debasing probabilities, so that

$$\begin{aligned} f_u &= (1+r)f_0 + q'S_0(u-d)h_0 && \text{if the underlying asset appreciates and} \\ f_d &= (1+r)f_0 - p'S_0(u-d)h_0 && \text{if the underlying asset depreciates} \end{aligned}$$

where $h_0 = (f_u - f_d)/(S_0u - S_0d)$, and f_u and f_d are the debased premiums calculated with p'. Letting Δ be the slack variable [between return and no return] and debasing constant $p' = p - \Delta$ and $q' = q + \Delta$ which decreases the probability of upsteps and increases the probability of downsteps; then the debased premium can be written in terms of the accurate probabilities by separating out the slack variable from the debased probabilities

$$\begin{aligned} f_u &= (1+r)f_0 + (q+\Delta)S_0(u-d)h_0 && \text{if the underlying asset appreciates and} \\ f_d &= (1+r)f_0 - (p-\Delta)S_0(u-d)h_0 && \text{if the underlying asset depreciates} \end{aligned}$$

So if the portfolio moves upwards, the asset appreciates to S_0u and the portfolio becomes Π_u

$$\begin{aligned} S_uh_0 - f_u &= S_0h_0u - (1+r)f_0 - (q+\Delta)S_0(u-d)h_0 \\ &= S_0h_0u - (1+r)f_0 - qS_0(u-d)h_0 - \Delta S_0(u-d)h_0 \\ &= (pu+qd)S_0h_0 - (1+r)f_0 - \Delta S_0(u-d)h_0 \end{aligned}$$

and if the portfolio moves downwards, the asset depreciates to S_0d and the portfolio becomes Π_d

$$\begin{aligned} S_dh_0 - f_d &= S_0h_0d - (1+r)f_0 + (p-\Delta)S_0(u-d)h_0 \\ &= S_0h_0d - (1+r)f_0 + pS_0(u-d)h_0 - \Delta S_0(u-d)h_0 \\ &= (pu+qd)S_0h_0 - (1+r)f_0 - \Delta S_0(u-d)h_0 \end{aligned}$$

which is, in both cases, an additional loss incurred on the short option $\Delta S_0(u-d)h_0$ because of the debased p (and that which debases the recursion for the accurate premium to that for the debased premium; similarly in continuous time). This is used to kill the return on the market asset to keep the portfolio return debased and flat at r, $\Pi_0 = v\Pi_1$. With

$$\Delta = \frac{pu + qd - 1 - r}{u - d}$$

for the asset debased with slack from probability, this loss incurred on the short option amounts to $(pu+qd-1-r)S_0h_0$ which is the market return on the market asset over the cash rate which is then sunk so that the portfolio return is then held flat at the cash rate for either the upstep $\Pi_0 = v\Pi_u$ or the downstep $\Pi_0 = v\Pi_d$. For $\Pi_0 = v\Pi_u$

$$\begin{aligned} S_u h_0 - f_u &= (pu+qd)S_0h_0 - (1+r)f_0 - S_0(u-d)h_0\frac{pu+qd-1-r}{u-d} \\ &= (pu+qd)S_0h_0 - (1+r)f_0 - (pu+qd-1-r)S_0h_0 \\ &= (1+r)S_0h_0 - (1+r)f_0 \\ &= (1+r)\Pi_0 \end{aligned}$$

rather than an interaction between the market return on the asset and the cash rate on the option. For $\Pi_0 = v\Pi_d$

$$\begin{aligned} S_d h_0 - f_d &= (pu+qd)S_0h_0 - (1+r)f_0 - S_0(u-d)h_0\frac{pu+qd-1-r}{u-d} \\ &= (pu+qd)S_0h_0 - (1+r)f_0 - (pu+qd-1-r)S_0h_0 \\ &= (1+r)S_0h_0 - (1+r)f_0 \\ &= (1+r)\Pi_0 \end{aligned}$$

which keeps the portfolio return fixed and flat at cash and void of market return [net of the update incurred but yet to be paid]. Sourcing the mispricing in the debased premium with returns earned on the market asset is what return-neutral valuation has misconstrued as "being true in every world", for which it is really meant that "the portfolio has no return no matter what the asset earns" since it is always killed by the mispricing in the premium. This has been the **justification for the debasement of derivatives** for **which risk-neutral valuation is synonymous and for which this PhD thesis is opposed**.

■

Proof 2

This proof corresponds one-for-one with that of PART II for the Gaussian basis function of continuous time. See page [insert here]. The delta-neutral replicating portfolio is

$$\Pi_0 = h_0S_0 - f_0$$

The recursion for the debased option premium *f '*, which is debased by varying the basis probability *p* by priming it with slack to $p' = p - \Delta$, is

$$f_0' = vp' f_u' + vq' f_d' \qquad f_0' = vp'f_u' + vq'f_d'$$

Debasing the portfolio with $v\Pi_1 = \Pi_0$ debases all items of the portfolio; the asset, the premium and the delta with the slack variable, in this case, where the slack variable is $\Delta = (pu+qd-1-r)/(u-d)$ taken from the upstep probability *p* (and added to the downstep probability *q*). The market asset *S*, the accurate premium *f*, and the delta of the accurate premium *h* are debased to the demeaned asset *Y*, the debased premium *f'* , and the debased delta *h'*. This is what return-neutral valuation calls *the alternative Q world that does not exist*; it is, rather, just a debased portfolio whereby each item of the portfolio is debased with the slack variable for a demeaned market asset *Y* and a premium and delta accurate for this demeaned asset rather than the market asset underlying the derivative. The portfolio is

$$\Pi_0 = h_0'Y_0 - f_0'$$

where the demeaned asset has debased basis components $Y_1 = Y_0u^{X'}d^{1-X'}$ for $X' \sim B(1,p') \sim B(1,p-\Delta)$. Remeaning the replicating asset by removing the slack variable from the demeaned asset but leaving the premium and delta debased with $p' = p - \Delta$ [removing the slack from the Bernoulli basis component $X' \sim B(1,p')$] (which decreases the variance of the asset whilst at the same time increasing the return profile to that of the market)

$$\Pi_0 = h_0'S_0 - f_0'$$

where the remeaned asset has basis components $S_1 = S_0u^Xd^{1-X}$ for $X \sim B(1,p)$. This is what return-neutral valuation calls *a return to the real P world*; it is, rather, a portfolio that remains debased and still earns nothing

despite the asset being remeaned. It is, instead, return-free in every circumstance. This is because the premium and deltas remain debased and the mispricing in the debased premium kills the return on the remeaned asset whatever this asset may then earn once it is remeaned by removing the slack from the Bernoulli basis components

$$\begin{aligned} E(\Pi_1 \mid \Pi_0) &= p(h_0' S_u - f_u') + q(h_0' S_d - f_d') \\ &= h_0'(pS_u + qS_d) - (pf_u' + qf_d') \\ &= (pu + qd)h_0' S_0 - (pf_u' + qf_d') \end{aligned}$$

Reintroducing the slack variable into the Bernoulli basis of the recursion for the option premium with the null additive term $0 = \Delta - \Delta$

$$X \sim B(1, p - \Delta + \Delta)$$

for the premium and then separating out one slack term to leave the debased basis component $X' \sim B(1, p - \Delta)$ so the recursion of the debased premium $f_0' = vp' f_u' + vq' f_d'$ can be utilized; the separated out slack term in Δ is then the additional change in the short option which kills the return on the market asset

$$\begin{aligned} pf_u' + qf_d' &= (p - \Delta + \Delta) f_u' + (q + \Delta - \Delta) f_d' \\ &= (p' + \Delta) f_u' + (q' - \Delta) f_d' \\ &= p' f_u' + q' f_d' + \Delta(f_u' - f_d') \\ &= p' f_u' + q' f_d' + \Delta(u - d) S_0 (f'_u - f_d')/(S_u - S_d) \\ &= p' f_u' + q' f_d' + \Delta(u - d) S_0 h_0' \end{aligned}$$

which is the recursion in the debased premium with the probabilities corresponding to the demeaned asset for which the premium is priced and for which the recursion applies $f_0' = vp' f_u' + vq' f_d'$. Note that the equivalent expression for the PDE of the debased premium in continuous time is

$rV - V_t = \frac{1}{2}\sigma^2 S^2 V_{SS} + rV_S = \frac{1}{2}\sigma^2 S^2 V_{SS} + (\mu + \frac{1}{2}\sigma^2) S V_s - \Delta S V_s$ where $\Delta = \mu + \frac{1}{2}\sigma^2 - r$ is the slack variable

and where the debasement term in the slack variable corresponds one-for-one. Hence

$$pf_u' + qf_d' = (1 + r) f_0' + \Delta(u - d) S_0 h_0'$$

and

$$E(\Pi_1 \mid \Pi_0) = (pu + qd)h_0' S_0 - \left((1 + r) f_0' + \Delta(u - d) S_0 h_0'\right)$$

where $\Delta = (pu + qd - 1 - r)/(u - d)$ so that the additional term for the change in the debased premium from time decay in the market asset rather than the demeaned asset is

$$\Delta(u - d) S_0 h_0' = \frac{pu + qd - 1 - r}{u - d}(u - d) S_0 h_0' = (pu + qd - 1 - r) S_0 h_0'$$

which should be compared with that for continuous time $(\mu + \frac{1}{2}\sigma^2 - r) S V_S'$ where $V_S' = h'$ is the option delta for the debased premium. The change in the option premium is therefore larger than the market rate, i.e., in excess of $pu + qd - 1$, since the value of the assets always exceeds the value of the option $S_0 h_0' > f_0'$. The portfolio then becomes

$$\begin{aligned} E(\Pi_1 \mid \Pi_0) &= (pu + qd)h_0' S_0 - \left((1 + r) f_0' + (pu + qd - 1 - r) S_0 h_0'\right) \\ &= (1 + r)h_0' S_0 - (1 + r) f_0' \\ &= (1 + r)\Pi_0 \end{aligned}$$

and the additional loss on the position in the short option has killed the return on the market asset and kept the replicating portfolio debased and null of the benefits of markets and free of convex flux in market return.

additionally show that portfolio variance is zero… variance *hS*, variance *f*, covariance of *hS* and f

Exactly the same principles apply for the portfolio and premium debased by varying the return factors with a slack variable, although the portfolio does not remain delta-neutral because of the variation in the discrete diffusion sample space upon removing the slack variable from the return factors rather than step probabilities [which

don't effect the sample space, but does mean nonsensically that the debased asset then is not correlated with the market since it then has a different relative frequency of upsteps]. For debased return factors, it is the same term that results from the debased premium upon remeaning the replicating asset, namely $(pu+qd-1-r)h_0S_0$, which similarly kills the return on the remeaned asset but there is an additional risky (binomial dependent) term because of the variation in the discrete sample space, and for example, an additional term in the deltas. The same sort of issues occur in continuous time where rebalances are made discretely.

Discuss going for zero NPV…must use market volatility… earn economic value of derivative (earn more for accurate premium; same intensity of trading… what people do anyway with implied volatility)… [theoretically] always a profit because of convex portfolio flux… not possible for multitude of reasons… stop writing white noise and discrediting people… allow accurate premiums [and research of accurate premiums] and revised philosophy for option markets to "best estimates of unbiased [accurate] premia".

■

With $f_j = v^{n-j}E\left(\max(Y_n-k,0)\,|\,S_j\right)$ and $f_n = \max(Y_n-k,0\,|\,S_n)$ for a call and $f_j = v^{n-j}E\left(\max(k-Y_n,0)\,|\,S_j\right)$ and $f_n = \max(k-Y_n,0\,|\,S_n)$ for a put, inclusive of **(1)** the delta update with a buy or sell in the market asset *S* rather than the demeaned asset *Y* and **(2)** time decay in the debased premium with respect to *S* rather than *Y*, the total portfolio change of the debased portfolio is

$$\Delta\Pi_j = r\Pi_j + S_{j+1}\Delta h_j$$

where the trade for the update $S_{j+1}\Delta h_j$ is statistically variable and is sourced from the rebalancer's cash at bank and consequently from balance sheet equity.

There are potential issues with replicating problems when returning the market return to the replicating asset whilst keeping the debased premium and deltas, in order for the market return to source the mispricing in the debased price and keep the portfolio return flat and debased at *r* [for a zero NPV of cashflow]. These potential issues regard replicating a premium priced for *Y* by trading the market asset *S* and deltas that correspond to the demeaned asset rather than the traded market asset, as well as the impossibility of maintaining an unbroken replication [and consequently an accumulation of unhedgeable gamma risk]. The debasement of the replicating portfolio is obtained by any method that sets $pu+qd=1+r$ for a flat portfolio return. For a portfolio and premium debased by varying *p* for the slack, as just considered, then the asset which subsidizes the mispricing has been remeaned [increasing to market return] whilst also having variance reduced so that it is a high signal-to-noise variable materially void of variation and contrary to any market variable for which more return is associated with more variance, not less. This therefore gives a false indication of the ability to source the mispricing competently with actual market returns since the sourcing asset miraculously has market return whilst having no market risk [which is ironically sourcing the debased premium with no market return but full market risk]. In actual markets, if you want more return you must take on more risk not less; for whatever reason, to source systematic mispricings or otherwise. An exponential market asset with all return and no risk is indeed an asset that does not exist! not in any real world spot market. If *u* is modified, then the remeaned asset is consistent with that of the markets for which the greater return from remeaning the asset that sources the mispricing in the debased option is associated with higher risk. The result is that the functions for the exercise price do not factor and do not adjust continuously to the modified return and has numerical downside issues which means that the remeaned asset does not competently source the mispricing in the debased option; this is symptomatic of **(1) an increase in variance with a change back to the market asset to replicate and source the mispricing in the debased option** [rather than sourcing it with a market asset void of statistical variation] which remains mispriced and accurate only for the demeaned asset and **(2)** variations in the discrete sample space and the problem associated with the exercise price and junk probabilities for the asset to deliver not being competently sourced by the market asset returns. This is in contrast to a modified probability for which the variance is decreased for the remeaning of the asset to source the mispricing in the debased premium. Whether it [the remeaned asset] does so [sources the mispricing] competently or not, it is still entirely pointless and anything but a valid excuse to debase derivatives and systematically misprice options.

(ii) Portfolio debased by letting the return factor *u* vary for the slack between return and no return

Proof 1

For the portfolio and premium debased by varying the parameter *u*, the recursion for the debased option premium is $f_0 = vpf_{u'} + vqf_{d'}$, where u' and d' are the return factors debased with slack Δ, so that

$f_{u'} = (1+r) f_0 + qS_0(u'-d')h_0$ if the underlying asset appreciates and
$f_{d'} = (1+r) f_0 - pS_0(u'-d')h_0$ if the underlying asset depreciates

where $h_0 = (f_{u'} - f_{d'})/(S_0u'-S_0d')$ is the delta of the debased option with respect to the debased asset with return factors *u′* and *d′***.** This delta for the debased premium with respect to the demeaned asset is smaller than that of the statistically accurate premium with respect to the market asset.

If the debased portfolio moves with an upstep with respect to the market asset *S* rather than the demeaned asset *Y*, the asset appreciates to $S_u = S_0u$ (rather than $Y_u = S_0u'$ for which the option is priced) and the portfolio becomes Π_u

$$S_uh_0 - f_u = S_0h_0u - f_u$$

but the recursion is for an upward movement in the debased asset $f_{u'}$ rather than an upward movement in the market asset f_u , i.e., the recursion for the premium priced for *Y* rather than *S*. So introducing $f_{u'}$ into the components of the portfolio for the upstep and substituting in the expression $f_{u'} = (1+r) f_0 + qS_0(u'-d')h_0$

$$\begin{aligned} S_uh_0 - f_u &= S_0h_0u - f_u \\ &= S_0h_0u - f_{u'} + (f_{u'} - f_u) \\ &= S_0h_0u - \left((1+r) f_0 + qS_0(u'-d')h_0\right) + (f_{u'} - f_u) \end{aligned}$$

The debased premium is

$$f_0 = v^n \sum_{x=0}^{n} \max\left(S_0(u')^x(d')^{n-x} - k, 0\right) {}^nC_x p^x q^{n-x} = v^n \sum_{x=a'}^{n} \left(S_0(u')^x(d')^{n-x} - k\right) {}^nC_x p^x q^{n-x}$$

where $a' = a'(S_0) = \log(k/S_0(d')^n)/\log(u'/d')$. Given the upstep movement in the market asset is with the actual upstep factor *u* rather than the debased factor *u′*, the premium becomes

$$f_u = v^{n-1} \sum_{x=0}^{n-1} \max\left(S_0u(u')^x(d')^{n-1-x} - k, 0\right) {}^{n-1}C_x p^x q^{n-1-x} = v^{n-1} \sum_{x=a'(S_0u)}^{n-1} \left(S_0u(u')^x(d')^{n-1-x} - k\right) {}^{n-1}C_x p^x q^{n-1-x}$$

with $a'(S_0u) = \log\left(k/S_0u(d')^{n-1}\right)/\log(u'/d')$ rather than

$$f_{u'} = v^{n-1} \sum_{x=0}^{n-1} \max\left(S_0u'(u')^x(d')^{n-1-x} - k, 0\right) {}^{n-1}C_x p^x q^{n-1-x} = v^{n-1} \sum_{x=a'(S_0u')}^{n-1} \left(S_0(u')^{x+1}(d')^{n-1-x} - k, 0\right) {}^{n-1}C_x p^x q^{n-1-x}$$

with $a'(S_0u') = \log\left(k/S_0u'(d')^{n-1}\right)/\log(u'/d') = \log\left(k/S_0(d')^n\right)/\log(u'/d') - 1 = a'(S_0) - 1 = a' - 1$. The difference between $a'(S_0u')$ and $a'(S_0u)$ is the term $-\log(\Delta)/\log(u'/d')$, so that this premium will have complications with competently sinking the return for the debased portfolio and premium since it does not adjust continuously for the variation in the nominalized exercise price for a change with respect to *S* rather than *Y*. In particular, there is slight numerical error in the discrete summation for every time the adjustment factor in a' for the max function causes a unit increment in one but not the other because of the difference between $a'(S_0u)$ and $a'(S_0u') = a'(S_0) - 1$. Ignoring the difference between $a'(S_0u)$ and $a'(S_0u')$

$$\begin{aligned} f_{u'} - f_u &= v^{n-1} \sum_{x=a'(S_0u')}^{n-1} \left(S_0(u')^{x+1}(d')^{n-1-x} - k\right) {}^{n-1}C_x p^x q^{n-1-x} - v^{n-1} \sum_{x=a'(S_0u)}^{n-1} \left(S_0u(u')^x(d')^{n-1-x} - k\right) {}^{n-1}C_x p^x q^{n-1-x} \\ &= v^{n-1} \sum_{x=a'(S_0u')}^{n-1} \left(S_0u'(u')^x(d')^{n-1-x} - S_0u(u')^x(d')^{n-1-x}\right) {}^{n-1}C_x p^x q^{n-1-x} \\ &= (u'-u)S_0 \sum_{x=a'(S_0u')}^{n-1} (vu')^x(vd')^{n-x-1} \, {}^{n-1}C_x p^x q^{n-1-x} \\ &= (u'-u)S_0h_0 \end{aligned}$$

and where the delta has an additional term because of the changed sample space of the lattice meaning that the exercise price does not remain on the lattice. This additional term due to the deltas is a further part to the amount incurred on the short option and differs depending on whether the step is up or down, meaning that **(1)** the portfolio does not remain delta neutral and **(2)** there is a slightly larger loss on the short option which eats into flat cash rate return (for $u = 1/d$) meaning that the replication will not competently source the mispricing in the debased premium. This is not trivial in this context and leads to a non-flat portfolio return net of the update (with returns below *r* depending on how it is debased) and statistical variation in the NPV of cashflow from a portfolio that does not competently offset delta. This is explored in more detail in an Addendum. Ignoring this numerical

issue for the time being [until the Addendum] for the sake of the argument and point at hand [sourcing the debased premium with market return], so that $f_{u'} - f_u = (u' - u)S_0 h_0$. Then the portfolio Π_u with $f_{u'} - f_u = (u' - u)S_0 h_0$ becomes

$$\begin{aligned} S_u h_0 - f_u &= S_0 h_0 u - \left((1+r) f_0 + qS_0(u' - d')h_0\right) + (u' - u)S_0 h_0 \\ &= S_0 h_0 u - (1+r) f_0 + S_0(pu' + qd')h_0 - S_0 u h_0 \\ &= (1+r)S_0 h_0 - (1+r) f_0 \end{aligned}$$

since $pu' + qd' = 1 + r$. The large increase in the short option premium $f_{u'} = (1+r) f_0 + S_0 u h_0 - (1+r)S_0 h_0$ has meant that the short position incurs a large loss within the portfolio that sinks the return on an upward step of the asset and keeps the portfolio return flat at the cash rate for $v\Pi_u = \Pi_0$ similarly to that debased by varying the parameter *p*.

Similarly for a downstep; if the debased portfolio moves with a downstep with respect to the market asset *S* rather than the demeaned asset *Y*, the asset depreciates to $S_d = S_0 d$ (rather than $Y_d = S_0 d'$ for which the option is priced) and the portfolio becomes Π_d

$$S_d h_0 - f_d = S_0 h_0 d - f_d$$

but the recursion is for a downward movement in the debased asset $f_{d'}$ rather than a downward movement in the market asset f_d . So introducing $f_{d'}$ into the components of the portfolio for the downstep and substituting in the expression $f_{d'} = (1+r) f_0 - pS_0(u' - d')h_0$

$$\begin{aligned} S_d h_0 - f_d &= S_0 h_0 d - f_d \\ &= S_0 h_0 d - f_{d'} + (f_{d'} - f_d) \\ &= S_0 h_0 d - \left((1+r) f_0 - pS_0(u' - d')h_0\right) + (f_{d'} - f_d) \end{aligned}$$

Ignoring the issues regarding the changed sample space of the lattice because of the debasement and deltas that correspond to *Y* rather than *S* but change with respect to *S* rather than *Y*, and a divergence between that of the market asset and that of the demeaned asset including the variance characteristics, issues that leave the portfolio not neutral to delta; then ignoring the difference between $a'(S_0 d)$ and $a'(S_0 d')$ and the additional term to the delta, similarly to the upstep, with the downstep at the actual factor *d* rather than the debased factor *d′*, the additional term is $f_{d'} - f_d = (d' - d)S_0 h_0$. The portfolio for a downstep Π_d with $f_{d'} - f_d = (d' - d)S_0 h_0$ then becomes

$$\begin{aligned} S_d h_0 - f_d &= S_0 h_0 d - \left((1+r) f_0 - pS_0(u' - d')h_0\right) + (d' - d)S_0 h_0 \\ &= S_0 h_0 d - (1+r) f_0 + S_0(pu' + qd')h_0 - dS_0 h_0 \\ &= (1+r)S_0 h_0 - (1+r) f_0 \end{aligned}$$

since $pu' + qd' = 1 + r$ and again, the portfolio return is then kept flat at the cash rate *r* for the debased portfolio $v\Pi_d = \Pi_0$. Similarly to that of a varied *p* to debase the portfolio and premium, in both cases of a step up and a step down, the additional loss incurred on the short option from the debased return factors is used to kill the market return to keep the portfolio return flat at *r*, and keep the debased portfolio change fixed flat at $\Pi_0 = v\Pi_1$.

■

Proof 2

The delta-neutral replicating portfolio is

$$\Pi_0 = h_0 S_0 - f_0$$

The recursion for the debased option premium *f ’*, which is debased by varying the return factors *u* and *d* by priming them with slack to u' and d', is

$$f_0' = vpf_{u'}' + vqf_{d'}'$$

where the subscripts in return ratios are primed rather than the coefficients in probabilities. Similarly to varying *p* with slack, debasing the portfolio with $v\Pi_1 = \Pi_0$ via step ratios debases all items of the portfolio with the slack variable; the asset, the premium and the delta. Depending on how the ratios are debased, this factor is the market return $\Delta = (puv + qdv)$ taken from both upstep and downstep ratios. The market asset *S*, the accurate premium *f*, and the delta of the accurate premium *h* are debased to the demeaned asset *Y*, the debased premium *f'* , and the debased delta *h'*. This is again the circumstance that return-neutral valuation calls *"the alternative Q world that does not exist"* which is simply a debased portfolio whereby each item of the portfolio is debased with the slack variable for a demeaned market asset *Y* and a premium and delta accurate for this demeaned asset rather than the market asset underlying the derivative, and for which all portfolio cashflows zero. The portfolio is

$$\Pi_0 = h_0' Y_0 - f_0' \qquad Y_1 = Y_0 (u')^{X'} (d')^{1-X'} = Y_0 (u'/d')^X (d')^n = Y_0 \exp\left(n \log(d') + X \log(u'/d')\right)$$

where the demeaned asset has debased basis components via the step ratios rather than step probabilities $Y_1 = Y_0 (u')^{X'} (d')^{1-X'}$ for $X \sim B(1, p)$. Remeaning the replicating asset back to *S* by removing the slack variable from the step ratios of the demeaned asset but leaving the premium and delta debased with u' and d' [removing the slack from the scaling of the steps] (for which the variance of return is increased and in the right direction with the increase in the mean return upon remeaning the asset from *Y* back to *S*)

$$\Pi_0 = h_0' S_0 - f_0'$$

where the remeaned asset has basis components $S_1 = S_0 u^X d^{1-X}$ for $X \sim B(1, p)$. This is similarly the circumstance that return-neutral valuation calls "*a return to the real P world*" whereby here the portfolio remains debased and still earns nothing despite the asset being remeaned. This is exactly analogous to continuous time. So what is meant by this, *"a return to the real world"*, is that the portfolio earns nothing in every circumstance, i.e., whatever the market asset earns. Similarly to that debased by varying *p*, this is because the premium and deltas remain debased and the mispricing in the debased premium kills the return on the remeaned asset whatever this asset may then earn once it is remeaned by removing the slack from the Bernoulli dependent step ratios

$$\begin{aligned} E\left(\Pi_1 \mid \Pi_0\right) &= p(h_0' S_u - f_u') + q(h_0' S_d - f_d') \\ &= h_0' (pS_u + qS_d) - (pf_u' + qf_d') \\ &= (pu + qd) h_0' S_0 - (pf_u' + qf_d') \end{aligned}$$

whereby the step ratios in the recursion for the premium are not primed since the asset for which they depend for the step is the remeaned asset, i.e., the expression is $pf_u' + qf_d'$ rather than $pf_{u'}' + qf_{d'}'$. But there is only an expression for $pf_{u'}' + qf_{d'}' = (1+r) f_0'$ to substitute into expression for the portfolio return and these step ratios are primed. So separating out the debasing slack for the forward step from $pf_{u'}' + qf_{d'}' = (1+r) f_0'$ for an expression in $pf_u' + qf_d'$ (free from slack in the subscripts but with debasing slack still in the ratios for all forward steps past this to the forward transition to maturity keeping the premium debased and inaccurate for *S*) which is then the recursion in the debased premium with the return ratios corresponding to the demeaned asset for which the premium is priced and for which the recursion $f_0' = vpf_{u'}' + vqf_{d'}'$ applies

$$\begin{aligned} pf_{u'}' + qf_{d'}' &= pf_{u'}' + qf_{d'}' \\ &= p(f_u' + f_{u'}' - f_u') + q(f_d' + f_{d'}' - f_d') \\ &= pf_u' + qf_d' + p(f_{u'}' - f_u') + q(f_{d'}' - f_d') \\ &= pf_u' + qf_d' - (pu + qd - r - 1) S_0 h_0' \end{aligned}$$

ignoring the replicating issues due to a changed walk sample space from removing the slack from the modified step ratios, and for a multitude of other reasons for not being able to keep the portfolio returns fixed flat and debased at *r*, assuming that there is no numerical error in $f_{u'}' - f_u' = (u' - u) S_0 h_0'$ and $f_{d'}' - f_d' = (d' - d) S_0 h_0$ so that

$$\begin{aligned} pf_{u'}' + qf_{d'}' &= pf_u' + qf_d' + p(u' - u) S_0 h_0' + q(d' - d) S_0 h_0' \\ &= pf_u' + qf_d' - (pu + qd - pu' - qd') S_0 h_0' \\ &= pf_u' + qf_d' - (pu + qd - 1 - r) S_0 h_0' \end{aligned}$$

hence

$$f_0' = vpf_{u'}' + vqf_{d'}'$$
$$= vpf_u' + vqf_d' - (puv + qdv - 1)S_0h_0'$$

i.e.,

$$pf_u' + qf_d' = (1+r)f_0' + (pu + qd - 1 - r)S_0h_0'$$

where the return factors for the one step transition [but not for the pricing] are free from slack and therefore depend for time decay on the remeaned asset so that the debased short option premium incurs a large additional loss $(pu+qd-1-r)S_0h_0'$ from depending on *S* rather than *Y* for which it is priced for that one step transition. This is exactly the term in the change in the premium with debased probabilities after the slack has been separated out, but this is most theoretically correct for $u' = u/\Delta$ and $d' = d/\Delta$ for $\Delta = (puv + qdv)$ and consistent with continuous time. The expectation of the portfolio change is then

$$E(\Pi_1 \mid \Pi_0) = (pu+qd)h_0'S_0 - \left((1+r)f_0' + (pu+qd-1-r)S_0h_0'\right)$$
$$= (1+r)h_0'S_0 - (1+r)f_0'$$
$$= (1+r)\Pi_0$$

The change in the option premium is therefore larger than the market rate, i.e., in excess of $pu+qd-1$, since the value of the assets always exceeds the value of the option $S_0h_0' > f_0'$, and the additional loss on the position in the short option has again killed the return on the market asset and kept the replicating portfolio debased and void of market return.

additionally show that portfolio variance is zero… variance *hS*, variance *f*, covariance of *hS* and f

■

Net of the time decay in the discount factor, the additional loss incurred on the short option is required to be **(i)** $S_0uh_0 - (1+r)S_0h_0$ for an upstep and **(ii)** $S_0dh_0 - (1+r)S_0h_0$ for a downstep to keep the portfolio return fixed and flat at *r*

$$h_0\Delta S_0 - rS_0h_0 = \begin{cases} S_0uh_0 - (1+r)S_0h_0 & \textit{for an upstep} \\ S_0dh_0 - (1+r)S_0h_0 & \textit{for a downstep} \end{cases}$$

which is larger than that of the accurate premium in order to amortize the mispricing and keep the portfolio debased and flat at *r*. The additional change of the accurate premium, in addition to the time decay in the discount factor, is **(i)** $S_0uh_0 - (pu+qd)S_0h_0$ for an upstep and **(ii)** $S_0dh_0 - (pu+qd)S_0h_0$ for a downstep. The larger loss on the short option with the debased premium is then $\Delta(u-d)S_0h_0 = (pu+qd-1-r)S_0h_0$ for either an upstep or a downstep. Therefore the additional term from the mispricing in the premium which sinks the return on the market asset is required to be $(pu+qd-1-r)S_0h_0$ if the mispricing is to be competently sourced by the earnings on the market asset and keep the portfolio return flat and debased at *r* since this is the term that is debased from the total derivative of the option premium, i.e., debased of $(pu+qd-1)S_0h_0$ and replaced by rS_0h_0. If there are any numerical issues which cause the additional loss on the short option to vary in any way from $(pu+qd-1-r)S_0h_0$ [because *f* is an option premium], then the portfolio deltas are not matched. This means that the return components are not kept flat at the cash rate for a flat portfolio return at *r* and the NPV of cashflow is not kept void of market variation and return.

In summary, the changes in the accurate premium are

(1) $f_u = (1+r)f_0 + S_0uh_0 - (pu+qd)S_0h_0$
(2) $f_d = (1+r)f_0 + S_0dh_0 - (pu+qd)S_0h_0$

which can be written as

(1) $f_u = (1+r)f_0 + S_0uh_0 - (1+r)S_0h_0 - (pu+qd-1-r)S_0h_0$
(2) $f_d = (1+r)f_0 + S_0dh_0 - (1+r)S_0h_0 - (pu+qd-1-r)S_0h_0$

When the portfolio and option are debased, then it is the term $(pu+qd-1-r)S_0h_0$ which is debased from the option, and the change in the debased premium is then

(1) $f_u=(1+r)f_0+S_0uh_0-(1+r)S_0h_0$
(2) $f_d=(1+r)f_0+S_0dh_0-(1+r)S_0h_0$

Then when removing the slack in the debased parameters of the replicating asset whilst keeping the option debased and mispriced, any issues that cause any variation in the change in the premium from **(1)** and **(2)** with respect to the actual (remeaned) market asset *S* rather than the demeaned asset *Y* [for which it is priced] so that it does not sink the market return that is being debased from the portfolio $(pu+qd-1-r)S_0h_0$ then leaves a portfolio change that is not free from binomial variation net of the update. In the context here, because the option premium does not adjust continuously as the **compounded slack in the premium** is removed with time decay from the actual market asset *S* rather than the demeaned asset *Y* for which the premium is accurate. The compounded slack in the premium is the mispricing in the premium and that for which is amortized from the premium to keep the portfolio debased of the benefits of the market asset for which the portfolio is trading, i.e., despite trading a market asset with return in excess of cash. Since the term debased from the portfolio and premium is proportional to the option delta, anything that impairs the ability of the compounded slack to be removed continuously from the premium to sink the returns of the traded asset means that the portfolio does not remain free from binomial variation and does not return a debased return fixed flat at cash [and ultimately results in a randomly variable {risky} non-zero NPV of cashflow {on either side of zero}].

This occurs as a problem **related to the fact** that the portfolio is not fully immunized from market dependent cashflow, rather than say amortizing a premium for the service of making a market in the derivative (say, for a future that recognizes return). These numerical issues do not affect the returns and cashflows for any node whereby all paths lead to either an inthemoney or an outofthemoney expiry, which is synonymous with any situation where an option is or becomes a pseudo-option which is either exercised with certainty or turned down with certainty, since the deltas then are deterministic functions that do not depend on ΔS; in which case the portfolio is debased of return by any method of debasing a premium to void market return and the portfolio is kept flat regardless [implying derivatives have no economic value]. Any time that it is an option proper that is debased by taking slack from *u*, any node for an option that includes paths that have both inthemoney and outofthemoney expiry values in the forward transition distribution results in a portfolio that is not free from delta and not flat at *r* due to the variation in the lattice and sample space [although this will always be dwarfed by voided replicating arguments from **gamma variation** due to an infinite number of missed rebalances]. This is also suggesting that to competently misprice an option premium to keep portfolio returns debased and flat for a zero NPV [no market return], traders would have better luck with low volatility market assets [although I still doubt that they will be able to keep their portfolio return flat at *r* for a zero NPV of portfolio cashflow].

■

Hence, for a portfolio and premium that are debased by taking slack from the return factors [which also limits to the debased premium of continuous time], when the numerical issues with sourcing the mispriced premium due to deltas of different distributions and samples spaces that do not adjust continuously are not overlooked, the total portfolio change is

$$\Delta\Pi_j=r\Pi_j+S_{j+1}\Delta h_j+c\Delta S_j$$

for some constant *c* for any option proper and where the binomial term results as the portfolio does not remain free from delta and ΔS because of the numerical issues. These numerical features leave the portfolio returns variable and dependent upon whether the market asset moves up or down [and hence not free from ΔS] and mean trading returns that are not sufficient to cover the bias. The probability distribution and the lattice of the sample space are fixed once the deltas and premiums have been calculated and this probability distribution and lattice of the various portfolio components corresponds to the demeaned asset and the demeaned asset only, i.e., the parameters of the debased portfolio with the demeaned asset are built into the premiums and deltas **and become fixed specific to, as well as unique to, the walk and portfolio lattice with those probabilities** – it is not then true "**in every world**" as risk-neutral tries to say for which it has misconstrued for subsidizing the mispriced premium with trading returns from a remeaned asset. As soon as the probabilities or return factors [which are one and the same in continuous time] are changed back by removing the slack variable to remean the asset after debasing the portfolio, the lattices of the deltas and premiums change back to those of the deltas and premiums of the statistically unbiased option; and if the premium is kept debased then it is the remeaned asset that attempts to subsidize the mispricing by sourcing it with trading returns to hold the portfolio return flat and debased of market return. The degree to which it can competently do this is questionable. A portfolio return that is not fixed and flat, resulting in a non-zero NPV of cashflow, is unequivocal evidence of not being able to debase

an option portfolio for a flat portfolio return by mispricing an option, i.e., unequivocal evidence of not being able to competently subsidize a mispriced premium with trading returns earned by flatlining the option for a cashflow void of return [entirely apart from the fact that doing so is entirely pointless]. The effect for modified return factors is that the replicating asset which is sourcing the mispriced premium does not properly replicate and offset the option when it is the actual asset that is traded for the debased premium and delta, and that does so with returns that are deficient of the cash rate (see Addendum which is for $u = 1/d$ and $u' = 1/d'$) and is, I believe [for good reason], like trying to put together a three piece jigsaw puzzle from two alternative puzzles, deltas and premiums for *Y* and a replicating asset from *S*. This is just another indication of problems consistent with not one incidence of the literature discussing successful delta-neutralized option portfolios returning a flat rate at *r* [and zero NPV of cashflow].

Yet another item of many that has the potential to invalidate any attempt at keeping the portfolio return flat at the cash rate by mispricing an option is the marketability of the bookvalue of the option premium. It is doubtful, for example, that a put option at the debased price is marketable in which case the flat portfolio return is a false measure of value and the neutralized delta position is again questionable; **if the premium is not marketable at the debased price, then the flat return at the cash rate is nothing more than a fudged accounting figure.** This is particularly so given that the literature admits time and time again that prices are not debased (Black and Scholes) prices and the issue of marketability of the bookvalues of the debased premium is just another one of many reasons that has overwhelming scope to invalidate the ability to debase an option portfolio for flat return [and zero NPV of cashflow] by mispricing a premium. Issues regarding the marketability of options on the secondary market were also admitted by Black and Scholes (1973) in addition to the admission of the debased price systematically deviating from market prices. Just the existence of the bid-ask spread on option prices necessarily means that this is a definitive issue that invalidates the ability to constrain a notional option portfolio flat at the cash rate by mispricing an option; i.e., a definitive divergence between the market value of the option (debased or otherwise) and the notional bookvalue within the portfolio. The replicating problems arise again when any rebalance and market fluctuation is skipped (any one of the *n* rebalances and trades to neutralize every market fluctuation for which delta is sufficient and only sufficient for binomial assets) including any missed rebalance after limits are taken whereby there are an infinite number of market fluctuations (and hence an infinite number $n \to \infty$ of required rebalances) over any non-infinitesimal period of time, as well as issues regarding not being able to neutralize gamma or any of the higher order effects. While it is possible to neutralize instantaneous delta, it is not possible in any circumstance to neutralize market risk for options proper since neutralizing delta is not sufficient, nothing close to it, which I have proved for multinomial walks and consequently any theoretically sensible market variable (multinomial basis with large *n*). **To rip the heart out of an option on a market variable [to flatline it] you need more than delta**. Here is the catch; only delta can be replicated [you cannot neutralize gamma] so it is impossible to go further than delta despite the fact that delta is insufficient. So even if you wanted to, as many have tried, subsidizing flatlined options with trading returns for a debased portfolio is impossible and fails always resulting in nothing but a skewed and convoluted option payoff profile of indeterminate form. Note that the only method to ascertain properly the cashflow paths of replicating the debased premium with the market asset is by the method of modifying the return factors to demean the asset. It is not possible with modified probability because it is probability that relates to the alternative paths and this probability is confounded along each and every path; once an event has happened, the probability is subjective as it only then has meaning in terms of relative frequency. It is dishonest to use an arbitrarily varied probability (say $p - \Delta$) of a past event to justify fudging an accounting figure and then saying that it doesn't matter since the event has already happened and the ex-post probability can be anything you want it to be; i.e., that the debased probability is another world that does not exist and that the probability for the event, now that it has happened, does not matter so that any accounting figure can be placed on its **ex-post expectation**. It should be reminded again that any one event [sample space] cannot have two probabilities [distributions]. It would be wrong to place a bookvalue of one million dollars for an expired lottery ticket because you have pretended that the ex-post probability of winning a two million dollar jackpot was one half. Risk-neutral valuation has regressed probability back several hundreds of years, and is worse than the error blundered by D'Alembert[18] (also regarding binomial probabilities). I doubt even people from the earliest days of probability theory would seriously consider saying that one variable could have two distributions at the same time, for example, that a die is fair whilst also being biased [or that the price of a fair game of chance should be skewed by pretending that it is biased]. A die is either fair or it is biased, it cannot be both. Similarly, a coin lands heads with probability *p* or it doesn't, it cannot land heads with both probability *p* and $p' = p - \Delta$. To say so is either ineptitude or dishonesty. Who are they trying to fool? themselves? Apart from the replicating issues, the method of debasing with *u* shows clearly what is happening

[18] This just goes to show that even clever people make mistakes, and none more so than in probability theory; when it comes to probability, I'm sure that many people would agree that it is best to err on the side of sensible.

and there is not the scope to be fooled into **believing nonsense such as 'alternative Q worlds that do not exist' and miraculous exponential market assets with little or no risk that can source systematic mispricings.**

The only possibility of any form of redemption for debased option premiums will be a documented [not faked in a fraudulent journal article] incidence of a zero NPV of cashflow of a debased premium, within a real derivatives market, for a portfolio trading a market asset from a real spot market. Until this time, debasing option portfolios holds absolutely no weight in the justification for debasing derivatives and systematically mispricing options. And even then, it will only have the vaguest possibility of utility as **a concept of killing everything in order to reduce variance**[19]. The onus of proof, which can only be made with **a documented case of zero NPV in actual markets**, is now on those whom are barring accurate prices and discrediting anyone who wants to be sensible by calculating [or researching] accurate premiums for options.

Anyone interested in pursuing risk-neutral valuation further (debasing portfolios and mispricing premiums), should stop writing white noise and discrediting people, and instead get busy and focus on trading in the markets and try to **get evidence of the ability to get a zero NPV of cashflow**[20]. Until they gather such evidence, if any, they should keep quiet and let people price their options accurately.

It is academically wrong to teach students that they are "free from risk" if they write a mispriced option and rebalance a couple of times. It is also wrong to teach students that they must take an offsetting position in the underlying asset and attempt to replicate, without telling them that they will then be [on the road to] killing their option [but failing]. To say that they have no risk if they do so is untruthful … this [no risk if the option is sold at half price[21]*] is nothing other than a gross extrapolation of nonsense.*

Flatlined Option Payoff Profiles [killed options]

Taking an offsetting position in the asset and replicating a debased option flatlines the payoff profile of the option, i.e., turns the payoff profile of the option into a flatline. What is meant by this is shown clearly diagrammatically for the discrete exponential binomial walk [which has diffusion sample space only on discrete points for S]

Long call



replicated long debased call



Long put



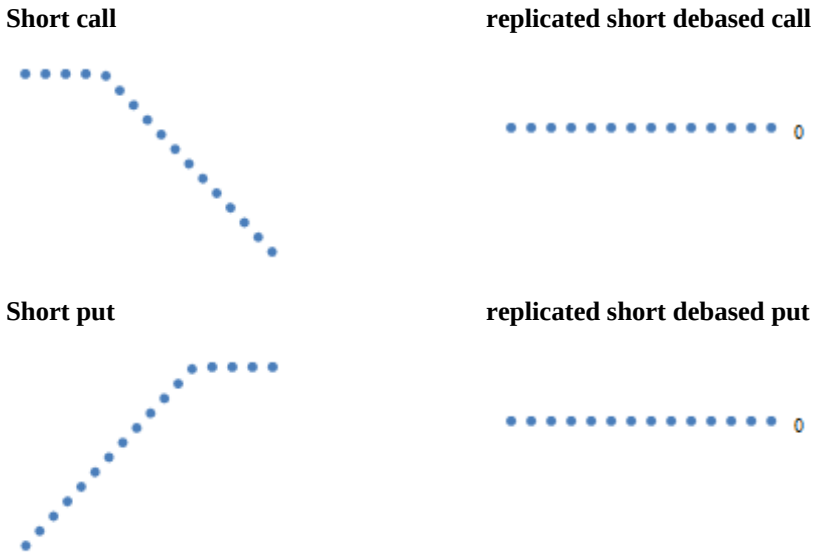
replicated long debased put



[19] and hence being more efficient to simply shut the option market down.

[20] I doubt they will get anywhere but I would be interested in reading about it [note that they will have to use market volatility rather than implied volatility]; all it is, as it stands, is people who should instead go and join their local communist party and be around people like themselves who believe that market returns, earnings, and profits should be banned.

[21] *without drift.*



If you want an option, you must not replicate and kill it. If all options must be replicated, in order to be subsidized [to subsidize the debased premium], as return-neutral says, then option payoff profiles should not be taught to students because they are then irrelevant. Further, the option should not be written in the first place and the option markets are better off closed down. It is then flatlined, successfully [which risk-neutral valuation fails to do], without having to do a hellofalota trading in the spot markets.

The NPV of Cashflows for the Debased Option Replicating Portfolio that Replicates with the Market Asset S

After debasing the portfolio of all return and demeaning S to Y (resulting in negative probabilities or probabilities greater than one if too much return has been removed in slack to debase the portfolio), the replication then tries to earn all the return debased from the premium back again by doing a whole lot of trading in S to finance the mispricing; where "doing a whole lot of trading" means a rebalance at each and every step and lattice node of the walk between writing the option and establishing the portfolio and closing out the portfolio at option expiry. **Every time the market moves, delta has to be rebalanced**. In particular, **whenever the market moves, matched deltas move out of parity and the replicating delta needs to be updated and re-matched to the option delta**. A replicating portfolio for an n-step expiry requires n trades for rebalancing the replicating delta for the offsetting asset to keep it matched with the delta of the option. This amounts to every single market fluctuation and includes continuous time limits to maintain the instantaneous delta for which the number of steps becomes infinite by making the step size small, in which case, amounts to sourcing a continuous stochastic signal {a Brownian motion} directly from equity. It is reminded that neutralizing delta only makes the portfolio free from ΔS for one instantaneous moment and if any single step (market fluctuation) is missed for a rebalance (or a single break in the continuous signal of instantaneous delta differentials), then the replication is broken and is consequently void and the end result is nothing other **than a skewed payoff profile** from accumulated gamma and higher order stochastic variation which accumulates from each and every rebalance that is missed. This is proved with multinomial walks and evidenced by a non-zero value of NPV of cashflow.

For the debased portfolio $\Delta\Pi_{j,n} = r\,\Pi_{j,n}$ and option $f_{j,n} = v^{n-j}E\left(\max(Y_n - k, 0) \mid S_j\right)$ replicated and offset by trading the demeaned asset S, over every step the premium is mispriced and undergoes large time value changes with respect to the market asset S rather than the demeaned asset Y for which it is accurate. These large changes in the debased premium then sink the returns earned from trading the market asset, killing the market return to keep the portfolio return flat and null at cash, and consequently zero the cashflow of the replication which nullifies the option. Overlooking potential replicating issues for using the remeaned asset whilst keeping the premium debased, the j^{th} change in the replicating portfolio is

$$\Delta\Pi_j = r\,\Pi_j + S_{j+1}\Delta h_j$$

inclusive of the update in S. Discounting and summing these n changes to the forward time variable n at option expiry, the net cashflow of the debased replicating portfolio is

$$v^n\,\Pi_n - \sum_{j=0}^{n-1} v^{j+1} S_{j+1}\Delta h_j - \Pi_0 = 0$$

where the term $\left(pu+qd-1-r\right)\sum_{j=0}^{n-1} v^{j+1}h_jS_j$ has gone entirely towards subsidizing the mispricing in the option premium, and the replication has again turned the option into a **redundant security void of a payoff profile.**

The zero net-present-value of the cashflows, under all paths, for a flatlined option results because of passing up all of the trading returns, returns that otherwise would have been earned from trading the excess returning asset in the hedge portfolio. Instead, these returns are used to subsidize the mispriced option which is flatlined as a result of the replication. **Any time that this ends up with a profit or a loss, rather than a zero NPV, the replicating arguments of the debased premium and portfolio have failed. As such, risk-neutral valuation always results in a failure [which is a good thing for options!].**

Overlooking the fact that this will always result in a failure and pretending that the end result is a zero NPV, given that it is the market asset *S* rather than the demeaned asset *Y* that is traded, the zeroed net cashflow is the result of the excess returns or losses on the period-to-period asset holdings accruing to the purchaser of the option rather than the hedge portfolio and all of the trading returns in going long and trading *S* being utilized to subsidize the mispricing in the option which is priced accurately only for *Y*. This trading may be large and is usually many times more than the value of the option itself, i.e., on an order of magnitude larger than that of the option. There is no benefit from the option since the replication turns it into a **redundant security void of a payoff profile,** nor benefit from the trading in the market asset *S* since all of the **market return is sunk by the mispriced option premium**. This is despite **(1)** trading within the environment of non-trivial market return linked with, and sourced from, economic growth and **(2)** making a market in the derivative by writing the security which defers the spot transaction and interest in the economic asset. The trading in the offsetting asset takes up this interest, so that the returns earned would ordinarily be the revenue for the financial service of making the market in the security that defers the spot transaction.

Since taking a position in an offsetting asset is expensive and kills the option, taking a position in and replicating with an offsetting asset should only attempted if you do not want the option. Any position in an offsetting asset only skews the payoff profile of the option and **the payoff profile you end up with is something between the call option and a covered call (or in the case of a put option, something between the profiles of the put option and a protected put)**. If you already have the two on your balance sheet (long market asset and short option or visa-versa), then it may be a good idea to try to arrange them (and possibly do a bit of additional trading) so that they balance each other [for offsetting deltas] to organize the balance sheet risk. Otherwise, if you want the option, you must not replicate with an offsetting asset and moreover, you should seek out call options and shun put options at debased prices for ordinary market conditions which then takes advantage of statistical mispricings in the debased option premiums. If you want to benefit from an option that you trade and want the payoff profile, then you must not replicate and kill it with an offsetting position.

Unfortunately for replicating portfolios, it is not possible to offset and neutralize every single market fluctuation and it is not possible to neutralize gamma or any degree higher than first, **nor is it possible to rebalance at, or anything close to, infinitesimals so it is impossible to maintain an unbroken replication from $\Pi_{0,n}$ to $\Pi_{n,n}$. One missed rebalanced [any market movement] means void replicating arguments and a skewed payoff profile of an indeterminate form from accumulations of gamma risk.** Once entered into the cashflows, gamma terms cannot be neutralized because of the inability to take positions in powers of the market asset. Such problems are many and arise from many sources, all of which invalidate entirely the ability to competently replicate an option to offset it, including sourcing mispricings in a debased premium with trading returns earned by flatlining it in order to keep the portfolio return flat and void of market return [i.e., zero NPV of cashflow].

Implied volatility [inefficiently] prices the slack debased from return back into the premium and delta i.e., inadvertently remeans the debased price. For call options, implied volatility should be larger than market volatility to price the slack taken from the mean back into the premium [and visa-versa for puts; smaller implied volatility to reduce the debased price]. So a debased premium calculated with implied volatility is implicitly a method to get an unbiased price, albeit an inefficient method. Anyone attempting to get a zero NPV with a debased portfolio and option premium should use market volatility rather than implied volatility, as doing so with implied volatility is implicitly replicating a statistically unbiased premium for *S* by trading *S* to the degree that implied volatility is accurate for taking up the numerical slack Δ debased from and biasing the premium [note that implied volatility is a function of term to maturity]. To get an accurate premium, the first consideration should be remeaning the debased price by adjusting the spot value for future return and using market volatility rather than implied volatility; using implied volatility for debased prices should only be considered as an inefficient[22] alternative or

[22] in the statistical sense of the word, i.e., inefficient because it is not minimum variance compared with adjusting the spot price; and additionally because implied volatility is not always defined since in some circumstances only returns a complex number which cannot sensibly be used as an input for a premium.

a secondary point of reference. Implied **return-volatility ellipses** can be used to determine under and overpriced premiums, or as a possibility, **implied return** from statistically unbiased prices equated to market prices with **market volatility for sigma**. It is essential to recognize return in derivatives, and it is wise to the use the option and derivative markets as a tool for analyzing return for trading decisions. The fact that prices are not debased (Black and Scholes), and never have been, is together with the volatility skew **evidence of return** in derivative markets and that the markets have a reversion away from being debased and flatlined. Moreover, this is evidencing that the derivative markets are simply an extension to the spot markets, rather than being redundant markets of debased and redundant securities.

A delta-neutral option replicating portfolio need not be debased so, if it were not for the multitude of problems associated with maintaining an unbroken replication, market participants and option traders could replicate anyway with the accurate premium and its replicating delta. This is essentially what is inadvertently [and inefficiently] happening when people replicate with a market asset for a debased premium and delta calculated with implied volatility rather than market volatility. Market participants should be allowed to earn a return for which they can then be certain that the replication is accurate for the market asset and a certain profit in the hold if they take the right side of the market for the replication, and in effect, swap the option for trading returns (which are on the same order of magnitude as the option) as revenue for the economic service provided by the derivative. So it is an assertion of this thesis that, in contrast to communism and trying to stamp out any return or profit [slack for which inadvertently just turns up somewhere else], market participants should be allowed to profit if they are smart enough to replicate, take the right side of the market, and get it right. In doing so, any revenue earned is simply that for the financial service of making a market in the derivative and taking up the pecuniary interest in the economic asset for the deferred transaction of the derivative. This profit is not infinite, but instead in direct proportion to the option for which the option writer is making a market in. This means that all hell will not break loose by recognizing return in derivatives [instead of trying to bar it], but instead there would just be one more good reason for market makers to be participating in the market. Given that the level of market return is never certain beforehand, it is not appropriate to say that the level of market return is not known, so we'll just pretend that there is no market return at all and give ourselves another excuse to keep derivatives debased. This is a cop out and you would not consider doing this with volatility. **Modeling stochastic variables on *S* is not easy. Technical analysis knows that it does not always get it right, but at least it tries and is motivated by the right reasons, namely, to earn return and profit from participating in the finances of the economy [rather than inadvertently trying to shut the market down] and help make markets efficient in the allocation of resources and finance, the medium by which the economy is powered.** Not a single person from finance would say that, because of the uncertainties involved, everyone should set beta to zero for every single security in the market for the Capital Asset Pricing Model (CAPM) so that all securities are priced as though the spot markets are not equity markets at all, and so that no one can benefit by participating in the market [since all sides, both long and short must then trade at the cash rate]. This is essentially what debasing option portfolios inadvertently tries to do, but all it ends up doing is making everything inaccurate and skewed. This is most notably evidenced by the volatility skew. As it stands, CAPM is a good start to remeaning debased premiums, simply by adjusting the spot price to account for future return, and better yet is that by direct reference to the option markets to get a price consistent with the market by observing market put-call parity [which it is **consistently observed that market put-call parity is not put-call parity of debased prices**]. It is the assertion of this thesis that prices should be accurate, returns should be allowed, and anyone who can make the most of statistical attributes of the dynamics of the market should be encouraged and praised. The market has quite a lot of experience with volatility with derivatives, it is suggested that this focus should also include return in derivatives as that which derives from the spot markets [rather than trying to stamp it out]. The derivative markets then provide another reference point to the spot markets, and in some regards, can be used as a **crystal ball** to see what the market believes for the future; **bull or a bear** indicated by market return and **certain or uncertain** indicated by market volatility.

Debased option portfolios of risk-neutral valuation are entirely inconsistent and incompatible with technical analysis and other forms of market analysis and consensus, including but not limited to CAPM. It is, rather, consistent with debased options as redundant securities and a completely flatlined option market [although failing always] whereby no person can benefit and instead everyone is made very busy trying to subsidize each other's option premiums by doing lots and lots and lots of trading, and whence giving each other their trading returns through mispricings in the option premiums for the redundant assets [options] for which they are flatlining and taking away and denying all market participants any incentive to trade options; hence options being redundant securities under the rationale of debased option portfolios. Subsidizing flatlined options with trading returns is so off the mark anyway that it is entirely defunct even if it wasn't a pointless circularity. Risk neutral valuation is a dysfunctional methodology for option markets; there is no purpose of an option on a bank account with full market risk except to create a redundant and useless financial security and a pointless financial market. Moreover it is wrong for anyone to tell someone who is clever enough to balance and immunize a portfolio that they

should not be allowed to make a small profit from a correctly judged position and from participating in the finances of economic activity and growth, and contributing to moving the economy forward. This thesis is presenting, therefore, an overwhelming case against the debasement of derivatives in favour of the sensible alternative of **striving to be accurate** in setting prices on securities [with probability theory and stochastic calculus on *S*] within **a derivative market that fully recognizes return** as a **fully complementary extension to the spot markets**.

Explaining the Statistical Bias in Debased Options in the Context of an Option Replicating Portfolio

This section compares the cashflows of the portfolios for the unbiased and statistically accurate premium with that of the debased portfolio and, by taking expected values, shows that the bias in the debased premium is almost entirely explained by the excess returns on trading the underlying asset in the cashflow equation of the accurate premium being moved into the initial premium. For the portfolio which has been debased by taking slack out of probability, the **bias** is a close subset of the **mean trading returns** that are earned with deltas of the accurate premium (i.e., the trading returns are slightly larger than the bias); whilst for the portfolio which has been debased by taking slack out of the return factors, the **mean trading returns** earned with the deltas of the accurate premium is a close subset of the **bias** (i.e., the trading returns are slightly smaller than the bias).

For a portfolio that has been debased by changing the binomial probability, the recursion for the premium is expressed as two parts by separating out the periodic mispricings in the premium with the slack variable; one **(1)** pertaining to the recursion that is updated by the replicating asset for which the debased option depends for time decay (rather than the demeaned asset for which it is priced) and **(2)** the other for the periodic mispricing for which is subsidized by the returns on the market asset. Then by solving the recurrence with the mispricing from the slack separated out, it is proved that the **bias** between the **accurate premium** and the **debased premium** is alternatively expressed as the **expected value of the trading returns** given up on the deltas of the debased option, which are used to sink the returns earned on trading the offsetting market asset to keep the portfolio return debased and flat at cash. This is also so for an option replicating portfolio and premium that have been debased by varying the parameters for the return factors, within numerical error because of **(1)** the discrepancy between the sample space of the demeaned asset *Y* for which the premium and deltas are accurate and that of the market asset *S* and **(2)** the increase in variance with a remeaning of *Y* to *S*. Because of the numerical issues pertaining to the additional term in the deltas from a changed lattice and an increase in variance in the remeaned asset, the portfolio does not remain delta neutral and the cashflow NPV is then randomly variable (and not zero). In this case, the trading returns earned on the remeaned asset used to flatline the option fall short of fully funding (subsidizing) the mispricing in the debased premium. Aside from this, the principles of sourcing the mispricing in the debased premium are the same as for modifying the parameter for the probability.

This thesis supports any mathematics that is valid and works, and is sensible for the context. The only measure of any utility at all of debasing an option premium [which as it stands is near nothing] is the ability to get a zero NPV of cashflow for the debased replicating portfolio; until someone can do this, which may not be ever, debasing an option portfolio and premium is simply a ***mathematical curiosity*** *and* ***not a basis to justify debasing derivatives and systematically mispricing options****.*

Comparison of Replicating Portfolios and the NPV of Cashflows

The debased option premium and delta are that for which the option price is accurate for the demeaned asset *Y*, the asset with all effective market return removed, as a result of debasing the option replicating portfolio with the constraint $v\Pi_{j+1} = \Pi_j$. For comparison purposes, the option premium and its delta priced for the demeaned probability distribution are denoted with a *debased* superscript. The debased option premia are

$f_{j,n}^{debased} = v^{n-j} E\left(\max(Y_n - k, 0) \mid S_j\right)$ for a call option and

$f_{j,n}^{debased} = v^{n-j} E\left(\max(k - Y_n, 0) \mid S_j\right)$ for a put option

where the backward variable is conditioned on the market spot price $Y_j = S_j$ and where

(1) $Y_n = S_j u^X d^{n-j-X}$ for $X \sim B(n-j, p-\Delta)$ for a demeaned variable by taking slack Δ out of p or

(2) (i) $Y_n = S_j (u/\Delta)^X (d/\Delta)^{n-j-X}$ for $X \sim B(n-j, p)$ for a demeaned variable by taking slack Δ out of u and d

(ii) $Y_n = S_j (u/\Delta)^X (d\Delta)^{n-j-X}$ for $X \sim B(n-j, p)$ for a demeaned variable by taking slack Δ out of u (u'=1/d')

These are accurate for the asset *Y* with no return for an $n-j$ step transition as conditional on the time *j* spot price $Y_j = S_j$ and consequently mispriced for on option on *S* for an $n-j$ step forward transition. The statistically accurate premiums for the call and put options for the market asset to deliver are

$f_{j,n} = v^{n-j} E\left(\max(S_n - k, 0) \mid S_j\right)$ for a call option and

$f_{j,n} = v^{n-j} E\left(\max(k - S_n, 0) \mid S_j\right)$ for a put option

where $S_n = S_j u^X d^{n-j-X}$ for $X \sim B(n-j, p)$. Only at option expiry at the forward variables are the accurate and debased premiums in parity. They differ for non-zero statistical bias at all other times, i.e., for any non-zero transition period; the longer the transition period and term to expiry, the larger the statistical bias and mispricing. The return components for the relevant replicating portfolios were proved to be

$\Delta\Pi_j = (pu + qd - 1) h_j S_j - r f_j$ for the accurate premium; and constrained to be

$\Delta\Pi_j^{debased} = r h_j^{debased} S_j - r f_j^{debased} = r\Pi_j^{debased}$ for risk-neutral valuation

net of the delta rebalance, one free to earn market return and the other debased of market return with a slack variable. The NPV of cashflow of the two replicating portfolios derive from summing the present value of these portfolio changes inclusive of the delta updates. This is an equation for the NPV of the cost of **(i)** establishing, **(ii)** rebalancing, and **(iii)** closing out the replicating portfolio. For the option priced accurately for the market asset *S* this is

$$v^n \Pi_{n,n} - \sum_{j=0}^{n-1} v^{j+1} S_{j+1} \Delta h_{j,n} - \Pi_{0,n} = \left(puv + qdv - 1\right) \sum_{j=0}^{n-1} v^j h_{j,n} S_j$$

market return on assets
(revenue for making a market in the derivative)

where $\Pi_{j,n} = h_{j,n} S_j - f_{j,n}$. For the debased portfolio for the option priced accurately for the demeaned asset *Y* but undergoes time decay in the market asset *S* and uses the market asset *S* to replicate and consequently expires for the market asset to deliver, with $\Pi_{j,n}^{debased} = h_{j,n}^{debased} S_j - f_{j,n}^{debased}$ the NPV of portfolio cashflows is

$$v^n \Pi_{n,n}^{debased} - \sum_{j=0}^{n-1} v^{j+1} S_{j+1} \Delta h_{j,n}^{debased} - \Pi_{0,n}^{debased} = 0$$

zero NPV of cashflow
(implies no economic value to derivatives)

ignoring the myriad of replicating issues[23]. The aggregate difference between the two sets of replicating cashflows is the right hand sides of the two equations, the first **(1)** for the statistically accurate premium containing the market return earned on the period-to-period holdings in the market asset and the second **(2)** being zero as a result of being debased. This difference for the NPV then arises solely because of the different premiums and deltas. The initial option premium is the only premium value that results in a transaction or cashflow and all other premium values are simply bookvalues until option expiry at which both expire at the same value because of successive conditioning on the backward variables whereby the derivative reverts to a spot transaction at the predetermined price *k* so that

$$f_{n,n}^{debased} = E\left(\max(Y_n - k, 0) \mid S_n\right) = \max(S_n - k, 0) = E\left(\max(S_n - k, 0) \mid S_n\right) = f_{n,n}$$

conditioned on the spot at maturity S_n for which is the only time the two premiums are in parity. Therefore, the only difference in cashflows apart from the differences in the initial premiums is from differences in the size of the deltas. Considering this fact and that market return on assets does not appear in the equation for the cashflows of the debased portfolio, moving the average of the return on assets into the initial accurate option premium explains almost all of the bias in the debased premium. The accumulated difference in the market return $puv + qdv - 1$ earned on the market asset *S*, for which is the return castrated from this asset for the demeaned asset *Y*, is used (imperfectly or otherwise) to subsidize the debased premium accurate only for the demeaned asset *Y* and keep the portfolio return flat at *r* and debased from the benefits of the market.

The debased premium is biased downwards and hence underprices calls on assets with positive market return and the opposite for put options. Calls are underpriced and puts are overpriced. Subtracting the terms in the expression for cashflows of the debased option from those of the statistically accurate option and noting that for both portfolios that trade the market asset *S*, then by conditioning at the forward variable step *n*, the closing out values are the same since the premiums and deltas are in parity for zero forward transition interval $(f_n - h_n S_n) - (f_n^{debased} - h_n^{debased} S_n) = 0$. Then

$$f_0 - f_0^{debased} = \left(puv + qdv - 1\right) \sum_{j=0}^{n-1} v^j h_j S_j + S_0 (h_0 - h_0^{debased}) + \sum_{j=0}^{n-1} v^{j+1} S_{j+1} (\Delta h_j - \Delta h_j^{debased})$$

[23] which invalidate entirely any hope of succeeding in debasing derivatives for a flatlined market.

for which it is seen that the only difference between the accurate premium and the debased premium apart from trading returns on the market asset is the differences in deltas. The remainder of the bias is spread throughout the returns earned on the difference between the magnitude of the deltas of the two portfolios. If there are any issues with the replication, then either of the two equations for the NPV of cashflows will be randomly variable and that for the debased portfolio will not be zero. Such issues arise whenever there is any question regarding how well the market asset subsidizes the mispriced premium, including any divergence between the market price and bookvalue of the option, and a multitude of other reasons. The expected value of the return on the differences in the deltas is the remainder of the bias resulting from the backed out premium from the debased portfolio. Taking expected values, for a call option

$$f_0 - f_0^{debased} = \left(puv + qdv - 1 \right) E\left(\sum_{j=0}^{n-1} v^j h_j S_j \right) + S_0 (h_0 - h_0^{debased}) + E\left(\sum_{j=0}^{n-1} v^{j+1} S_{j+1} (\Delta h_j - \Delta h_j^{debased}) \right)$$

The large majority of the trading returns for $p = 0.80$ comprise the bias so the return on the market assets is the dominant term (for $p = 0.80$) explaining almost all of the bias in the debased premium. The remainder is return from the deltas and delta differences. For modified probabilities, the remaining difference is due to returns and returns on returns of the larger deltas (for a call) for the statistically accurate premium, and hence slightly larger positions in the market asset and hence slightly larger returns over and above the bias, i.e., the mean value of the trading returns of the unbiased and accurate portfolio is larger than the bias from the option backed out of the debased portfolio. For modified return factors, the trading returns earned on the deltas are smaller than the bias and are insufficient to cover the bias and mispricing due to an increase in variance upon remeaning the replicating asset.

An exact expression for the bias is obtained when the differences in deltas are taken into account for which the expected value of the returns earned on the market asset is that which is used to subsidize the mispriced option, so that the mispricing sinks the market return as it is earned to keep the option portfolio flat at *r* and debased of market return. The definition of the bias is the difference between the accurate and debased premiums; this then provides for another formula for the exact bias and is considered now. It is shown that the statistical bias in the debased price is the expected value of the trading returns passed up on trading the market asset with the deltas of the debased premium. This is proved in the case where the portfolio is debased by taking slack from the parameter *p*. When the portfolio is debased by taking slack from the return factors, the subsidization of the mispriced premium [by flatlining it] has numerical issues as any replicating subsidization will have in any practical setting. The numerical error arises from a changed sample space and increase in variance resulting from a return back to the return profile of the market [whilst keeping the portfolio and premium debased] to subsidize the mispricing and a greater rate of change in the debased option. This flags some of the numerous problems of trying to do this in practice to keep a portfolio debased by mispricing an option. Overlooking the numerical issues, the bias as the expected value of trading returns earned from flatlining the option and given up to subsidize the mispriced premium is also proved for debased return factors showing that the principles of debasing by taking slack from *u* exactly parallels those by debasing by taking slack from *p*. Apart from these issues, debasing with *u* is more theoretically sensible and has less scope for extrapolating nonsense [like alternative worlds that don't exist] than debasing with *p*. Additionally, this hints more clearly that it is not possible to do this [subsidize flatlined options with trading returns for flat debased portfolio return and zero NPV] in practice with continuously varying market variables, which can only be traded at discrete points in time, since premiums debased by varying either *p* or *u* with debasing slack both limit to the debased price in continuous time.

Explaining the Vanished Mean and the Bias is the Debased Premium

Recalling that the market asset traded within the debased portfolio stills earns market return, independently of any mispricing in the option; the mispricing in the premium is sourced by sinking the market return on the offsetting asset as it is earned whilst maintaining delta throughout the forward transition to maturity in order to kill the delta of the option. This return earned on the market assets that is used to subsidize the mispriced premium is

$$\left(puv + qdv - 1 \right) \sum_{j=0}^{n-1} v^j S_j h_j^{debased}$$

If this is not used to subsidize the mispriced premium then this would be the term of the NPV cashflow equation for return on assets. Adding the term back to both sides of the cashflow equation (or alternatively not canceling it before summing the portfolio changes)

$$v^n \Pi_n^{debased} - \sum_{j=0}^{n-1} v^{j+1} Y_{j+1} \Delta h_j^{debased} - \Pi_0^{debased} + \left(puv + qdv - 1 \right) \sum_{j=0}^{n-1} v^j h_j^{debased} S_j = \left(puv + qdv - 1 \right) \sum_{j=0}^{n-1} v^j h_j^{debased} S_j$$

and shifting it into the initial premium so that

$$v^n \Pi_n^{debased} - \sum_{j=0}^{n-1} v^{j+1} Y_{j+1} \Delta h_j^{debased} - \left(h_j^{debased} S_0 - f_0 \right) = \left(puv + qdv - 1 \right) \sum_{j=0}^{n-1} v^j h_j^{debased} S_j$$

with $f_0 = f_0^{debased} + \left(puv + qdv - 1 \right) \sum_{j=0}^{n-1} v^j h_j^{debased} S_j$. This is a correct form of the cashflow equation for the market asset that earns $puv + qdv - 1$ in excess of the cash rate and where, for a call, $h_j^{debased} < h_j$. Ordinarily, the return is contained within the deltas but is earned on the asset which gives further suggestion that the premium, its delta and the replicating asset are three parts to one jigsaw set. Ironically here, the debased deltas have no return but the asset still earns return! In the context of subsidizing a mispriced premium [and consequently killing the option], this again calls into question the ability of a return to the 'real world' whilst keeping the debased price; with correct logic, this is whether or not the remeaned asset can competently source the mispricing in the debased premium given that none of the *n* mandatory rebalances for an *n* step ahead expiry are missed to keep the portfolio return flat and debased at *r* and keep the NPV of cashflow zero and debased of return.

It is now proved that by taking expected values, the mean value of this term is the bias between the debased premium and the accurate premium for the market asset to deliver

$$f_0 - f_0^{debased} = Bias = \left(puv + qdv - 1 \right) E\left(\sum_{j=0}^{n-1} v^j S_j h_j^{debased} \mid S_0 \right) > 0 \quad \text{for a call option}$$

$$f_0 - f_0^{debased} = Bias = -\left(puv + qdv - 1 \right) E\left(\sum_{j=0}^{n-1} v^j S_j h_j^{debased} \mid S_0 \right) < 0 \quad \text{for a put option (with a sign change in } h\text{)}$$

so that the bias between the accurate premium and the debased premium is equal to the expected value of the return earned on the market asset traded within the debased portfolio at the deltas of the debased premium; a call is underpriced and a put is overpriced by the debased premium whenever $pu + pd > 1 + r$; the signs are reversed whenever $1 + r > pu + pd$ so that a call is overpriced and a put is underpriced by the debased premium in bear markets. Given the extent to which the replicating asset can accurately source the mispricing in the debased premium and accurately sink the trading returns of the market asset rather than the demeaned asset for which it is priced is the extent to which this formula [mean of trading returns with debased deltas] is accurate as the difference between the premiums and the statistical bias of the debased price. This is what risk-neutral has misconstrued as a 'return to the real world' whilst keeping the debased premium so that the mispriced premium sinks the trading returns and the replication [that kills the option] earns the returns to misprice it. Overlooking any issues with the replication to source the debased premium, for the expectation to be accurate, among other things, there must be no missed rebalances, i.e., a total of *n* rebalances for an option that expires in *n* steps.

(i) The premium debased by varying the probabilities *p*′ and *q*′: Sourcing the mispricing with trading returns earned by flatlining the option

For the statistically accurate premium and replicating portfolio that has not been debased, the backwards recursion for the premium is $f_0 = vpf_u + vqf_d$. It was shown that solving $f_0 = vpf_u + vqf_d$ for the boundary conditions returns the statistically accurate premium f_0. Debasing the portfolio by reducing *p* and increasing *q* with slack Δ, to $p' = p - \Delta$ and $q' = q + \Delta$, the recursion for the debased premium is then that for the accurate recursion but with the debasing probabilities $f_0^{debased} = vp' f_u^{debased} + vq' f_d^{debased}$ which suppose that upsteps are less likely and downsteps are more likely so that the premium is void of market return. Solving $f_0^{debased} = vp' f_u^{debased} + vq' f_d^{debased}$ whilst keeping the probabilities varied with slack for the boundary conditions returns the debased premium $f_0^{debased}$. Expressing the recursion for the debased premium with the accurate probabilities by separating out the mispricing from the junk probabilities $p' = p - \Delta$ and $q' = q + \Delta$ so that the debased premium has time decay in the market asset [which is remeaned with an decrease in variance] rather than the demeaned asset, the recursion can be written as

$$\begin{aligned} f_0^{debased} &= vp' f_u^{debased} + vq' f_d^{debased} \\ &= v(p - \Delta) f_u^{debased} + v(q + \Delta) f_d^{debased} \\ &= vpf_u^{debased} + vqf_d^{debased} - \Delta v(f_u^{debased} - f_d^{debased}) \end{aligned}$$

This is the recursion in the correct and accurate probabilities, rather than junk probabilities, less the amount that is sunk at each step of the walk from the mispricing in the debased premium. Noting that the slack variable Δ taken from *p* which debases the portfolio is

$$\Delta = \frac{pu + qd - 1 - r}{u - d}$$

so the recursion can then be written as

$$f_0^{debased} = vpf_u^{debased} + vqf_d^{debased} - (puv + qdv - 1)S_0 h_0^{debased}$$

where $h_0^{debased}$ is the delta of the debased premium

$$h_0^{debased} = \frac{f_u^{debased} - f_d^{debased}}{S_0 u - S_0 d}$$

and is hence always smaller than the delta of the accurate premium $h_0^{debased} < h_0$. The term

$$(puv + qdv - 1)S_0 h_0^{debased}$$

is the periodic mispricing in the debased premium, which sinks the return on the market asset within the debased portfolio to keep the portfolio return flat at the cash rate $v\Pi_1 = \Pi_0$. For the next step in the walk

$$\begin{aligned} f_u^{debased} &= vp' f_{uu}^{debased} + vq' f_{ud}^{debased} \\ &= vpf_{uu}^{debased} + vqf_{ud}^{debased} - (puv + qdv - 1)S_u h_u^{debased} \end{aligned} \quad \text{if the first step in } S \text{ is to } S_0 u \text{ (with frequency } p)$$

$$\begin{aligned} f_d^{debased} &= vp' f_{du}^{debased} + vq' f_{dd}^{debased} \\ &= vpf_{du}^{debased} + vqf_{dd}^{debased} - (puv + qdv - 1)S_d h_d^{debased} \end{aligned} \quad \text{if the first step in } S \text{ is to } S_0 d \text{ (with frequency } q)$$

where the periodic mispricing is $(puv + qdv - 1)S_u h_u^{debased}$ or $(puv + qdv - 1)S_d h_d^{debased}$ depending on whether the spot price moved up or moved down, and similarly sinks the return on the market asset for the second step keeping the portfolio which trades the market asset debased and flat at the cash rate $v\Pi_2 = \Pi_1$. For the third step

$$\begin{aligned} f_{uu}^{debased} &= vpf_{uuu}^{debased} + vqf_{uud}^{debased} - (puv + qdv - 1)S_{uu} h_{uu}^{debased} && \text{if the second step in } S \text{ is to } S_0 u^2 \\ f_{ud}^{debased} &= vpf_{udu}^{debased} + vqf_{udd}^{debased} - (puv + qdv - 1)S_{ud} h_{ud}^{debased} && \text{if the second step in } S \text{ is to } S_0 ud \\ f_{dd}^{debased} &= vpf_{ddu}^{debased} + vqf_{ddd}^{debased} - (puv + qdv - 1)S_{dd} h_{dd}^{debased} && \text{if the second step in } S \text{ is to } S_0 d^2 \end{aligned}$$

for $v\Pi_3 = \Pi_2$ and similarly for all steps to maturity at the forward variables at step *n*. So the backward recursion in the debased premium $f_0^{debased} = vp' f_u^{debased} + vq' f_d^{debased}$ can then be written and solved with respect to the correct distribution of *S* by separating out the periodic mispricings in the debased premium which sink the market return of the replicating asset and hold the return on the portfolio flat at the cash rate and debased from the benefits of the spot markets

$$\begin{aligned} f_0^{debased} &= vpf_u^{debased} + vqf_d^{debased} - (puv + qdv - 1)S_0 h_0^{debased} \\ &= vp\left(vpf_{uu}^{debased} + vqf_{ud}^{debased} - (puv + qdv - 1)S_u h_u^{debased}\right) + vq\left(vpf_{ud}^{debased} + vqf_{dd}^{debased} - (puv + qdv - 1)S_d h_d^{debased}\right) \\ &\qquad -(puv + qdv - 1)S_0 h_0^{debased} \\ &= v^2 p^2 f_{uu}^{debased} + 2v^2 pqf_{ud}^{debased} + v^2 q^2 f_{dd}^{debased} \\ &\qquad -vp(puv + qdv - 1)S_u h_u^{debased} - vp(puv + qdv - 1)S_d h_d^{debased} - (puv + qdv - 1)S_0 h_0^{debased} \\ &= v^2 p^2 f_{uu}^{debased} + 2v^2 pqf_{ud}^{debased} + v^2 q^2 f_{dd}^{debased} \\ &\qquad -(puv + qdv - 1)\left(vpS_u h_u^{debased} + vpS_d h_d^{debased} + S_0 h_0^{debased}\right) \\ &= v^2 p^2 \left(vpf_{uuu}^{debased} + vqf_{uud}^{debased} - (puv + qdv - 1)S_{uu} h_{uu}^{debased}\right) + 2v^2 pq\left(vpf_{uud}^{debased} + vqf_{udd}^{debased} - (puv + qdv - 1)S_{ud} h_{ud}^{debased}\right) \\ &\qquad +v^2 q^2 \left(vpf_{udd}^{debased} + vqf_{ddd}^{debased} - (puv + qdv - 1)S_{dd} h_{dd}^{debased}\right) \\ &\qquad -(puv + qdv - 1)\left(vpS_u h_u^{debased} + vpS_d h_d^{debased} + S_0 h_0^{debased}\right) \\ &= v^3 p^3 f_{uuu}^{debased} + 3v^3 p^2 qf_{uud}^{debased} + 3v^3 pq^2 f_{udd}^{debased} + v^3 q^3 f_{ddd}^{debased} \\ &\quad -(puv + qdv - 1)\left(v^2 p^2 S_{uu} h_{uu}^{debased} + 2v^2 pqS_{ud} h_{ud}^{debased} + v^2 q^2 S_{dd} h_{dd}^{debased} + vpS_u h_u^{debased} + vpS_d h_d^{debased} + S_0 h_0^{debased}\right) \\ &\ldots \\ &= v^n \sum_{x=0}^{n} {}^n C_x p^x q^{n-x} f_{u^x d^{n-x}}^{debased} - (puv + qdv - 1)\sum_{j=0}^{n-1} v^j \sum_{x=0}^{j} {}^j C_x p^x q^{j-x} S_{u^x d^{j-x}} h_{u^x d^{j-x}}^{debased} \\ &= f_0 - (puv + qdv - 1)\sum_{j=0}^{n-1} v^j E\left(S_j h_j^{debased} \mid S_0\right) \end{aligned}$$

The debased premium and the premium accurate for *S* are in parity for zero transition period at, and only at, option expiration at the forward variables at step *n* so $f_{u^x d^{n-x}} = f^{debased}_{u^x d^{n-x}}$ since the debased premium and accurate premium expire at the same boundary when both are replicated by, and successively conditioned on, the market asset *S*. An *n* step option maturity means *n* rebalances. The term

$$f_0 = v^n \sum_{x=0}^{n} {}^nC_x p^x q^{n-x} f^{debased}_{u^x d^{n-x}} = v^n \sum_{x=0}^{n} {}^nC_x p^x q^{n-x} f_{u^x d^{n-x}}$$

is the premium calculated at the correct probabilities since the slack Δ which debased the premium has been separated out, and is hence the accurate premium f_0 for the market asset to deliver. The premium of the debased price is

$$f^{debased}_0 = v^n \sum_{x=0}^{n} {}^nC_x (p')^x (q')^{n-x} f^{debased}_{u^x d^{n-x}} = v^n \sum_{x=0}^{n} {}^nC_x (p')^x (q')^{n-x} f_{u^x d^{n-x}}$$

The amount of return that has been sunk from the replicating market asset in order to keep the portfolio flat is **(1)** the present value of the sum of the periodic mispricings from the slack Δ as well as **(2)** the difference between the two premiums, the accurate premium and the debased premium. The double summation

$$(puv + qdv - 1) \sum_{j=0}^{n-1} v^j \sum_{x=0}^{j} {}^jC_x p^x q^{j-x} S_{u^x d^{j-x}} h^{debased}_{u^x d^{j-x}}$$

is the expected present value of the sum of the periodic mispricings. It was shown that solving $f_0 = vpf_u + vqf_d$ for the boundary conditions returns the statistically accurate premium f_0. Similarly solving $f^{debased}_0 = vp' f^{debased}_u + vq' f^{debased}_d$ whilst keeping the fake debasing probabilities [rather than returning to the accurate probabilities by separating out the mispricing] for the boundary conditions returns the debased premium $f^{debased}_0$. Hence the difference between the accurate premium and the debased premium is

$$\begin{aligned} f_0 - f^{debased}_0 &= (puv + qdv - 1) \sum_{j=0}^{n-1} v^j \sum_{x=0}^{j} {}^jC_x p^x q^{j-x} S_{u^x d^{j-x}} h^{debased}_{u^x d^{j-x}} \\ &= (puv + qdv - 1) \sum_{j=0}^{n-1} v^x E\left(S_j h^{debased}_j \mid S_0 \right) \end{aligned}$$

where the outer summation is the sum over the path of delta positions, over the backward time variable *j*, with respect to the remeaned asset with the accurate probabilities *p* and *q* rather than the probabilities debased with slack $p' = p - \Delta$ and $q' = q + \Delta$. The inner summation is over the forward variable *x* of the walk at option expiration at the forward time variable *n* and is hence an expectation with respect to the forward walk transition distribution. Hence, this thesis has proved that the difference between the debased price $f^{debased}_0$ and the statistically accurate price f_0 is the expected value of the return on assets of the replication where the deltas are those for the debased premium $h^{debased}_j$ [smaller than those for the accurate premium which recognizes return] for which have gone entirely towards sinking the market return on the replicating market asset. Hence, the expected value of the returns passed up on trading *S* is the difference between the two premiums

$$Bias = (puv + qdv - 1) \sum_{j=0}^{n-1} v^x E\left(S_j h^{debased}_j \mid S_0 \right) > 0$$

similarly for a put option for which a debased premium overprices the option

$$Bias = f_0 - f^{debased}_0 = -(puv + qdv - 1) \sum_{j=0}^{n-1} v^x E\left(S_j h^{debased}_j \mid S_0 \right) < 0$$

This is the statistical bias and inaccuracy in the debased premium, and to whence the return has disappeared from the premium; it is debased from the portfolio and premium and then sourced back to subsidize a premium that remains debased from returns earned from trading the market asset which replicates, offsets, and flatlines the option. **No trading returns and no option**. This is what this book has set out to explain in order to make the case for accurate premiums which is unequivocal. This expected value and difference in premiums is an identity and exact (including on a spreadsheet). An option with *n* steps to maturity means *n* total trades in the market asset to replicate the premium and the option otherwise the expected value will be a biased measure of the bias and this equation for the bias will be invalid.

In the context of a spreadsheet, then, **chain back with the proper probability (and return factors) using the recursion for the accurate premium** $f_0 = vpf_u + vqf_d$ **rather than those for a debased premium** $f^{debased}_0 = vp' f^{debased}_u + vq' f^{debased}_d$ **or** $f^{debased}_0 = vpf^{debased}_{u'} + vqf^{debased}_{d'}$. The portfolio then earns the market return on

the market asset and incurs the cash rate on the short option. From page 24, the resulting return profile is **(1)** the economic value of the deferred transaction of the derivative and **(2)** the value of the financial service of making a market in the derivative and taking up the pecuniary interest in the deferred transaction. Using probabilities or return factors debased with slack debases everything of market return, as just proved; the debased premium sinks the return on the asset from the statistical mispricing in the premium from the slack variable. Debasing a portfolio of return by mispricing a premium at a debased price is consistent only with stupidity or communism [or both]. It is time to move on.

■

When the portfolio and premium are debased by varying the return factors, there is a numerical discrepancy because of the variation in the sample space and an increase in variance with a return back to the return profile of the market asset. The principle of debasing the asset is, apart from this, entirely analogous; nice maths but in no way suited to the financial markets. If the numerical issues pertaining to the changed sample space are overlooked [deltas with an additional term and the minimum number of upsteps having some cumulative error], then the method works similarly for debasing the portfolio by varying the return factors to show that the statistical bias in the debased price is the expected value of trading returns passed up on trading the market asset to fund the mispricing in the debased portfolio and premium to keep the portfolio return fixed flat and debased. It is therefore proved that it is not another probability world that has no risk and does not exist and is true in every world; it is, instead, just a debased and flatlined option.

(ii) The premium debased with return factors u' and d': Sourcing the mispricing with trading returns earned by flatlining the option

When the replicating portfolio is debased by reducing u and d, to $u' = u/\Delta$ and $d' = \Delta/d$, the recursion for the debased premium is then that for the accurate premium with the debased return factors $f_0^{debased} = vpf_{u'}^{debased} + vqf_{d'}^{debased}$ making upsteps smaller and downsteps larger so that upsteps earn less and downsteps incur more for a debased premium that is priced for a demeaned variable that is void of market return. This recursion differs from that of varying the parameter p, and instead has u and d primed rather than p and q which feature in the recursion in the **primed subscripts rather than in the primed coefficients**, and in the boundary condition in the max function rather than the step generator. The approach, therefore, proceeds slightly differently to source the mispricing in the debased premium with returns from the offsetting market asset.

The backwards recursion for the statistically accurate premium is $f_0 = vpf_u + vqf_d$ which solves with the relevant boundary conditions for the statistically accurate premium f_0. Similarly solving the recursion for the debased price $f_0^{debased} = vpf_{u'}^{debased} + vqf_{d'}^{debased}$ returns the debased premium $f_0^{debased}$. The debased returns are now separated out from the recursion so that it is split into two parts; a component for which the actual return factors apply for the time decay in the option and a component which pertains to periodic mispricings for which sink the returns on the market asset to keep the debased portfolio return flat at the cash rate.

Separating out the mispricing from the debased return factors, noting that $u' = u/\Delta$ and $d' = d/\Delta$, the recursion can be written as

$$\begin{aligned} f_0^{debased} &= vpf_{u'}^{debased} + vqf_{d'}^{debased} \\ &= vp(f_u^{debased} - f_u^{debased} + f_{u'}^{debased}) + vq(f_d^{debased} - f_d^{debased} + f_{d'}^{debased}) \\ &= vpf_u^{debased} + vqf_d^{debased} - \left(vp(f_u^{debased} - f_{u'}^{debased}) + vq(f_d^{debased} - f_{d'}^{debased})\right) \end{aligned}$$

given that the market asset is traded and the premium incurs time value changes at a rate dependent upon the actual returns of the market asset S. This amounts to larger changes and losses in time value for the short option as a result of the larger return of the market asset than that priced into the premium. This then sinks the return on the market asset that is used to replicate and kill the option. Ignoring the numerical issues related to the small issue with changes in *a'* for the minimum number of upsteps for the option to expire inthemoney and that related to an additional term in the delta as a result of the changed sample space, both of which arise from an increase in variation with a remeaning of the asset, then as shown previously

$$f_u - f_{u'} = (u - u')S_0 h_0^{debased}$$
$$f_d - f_{d'} = (d - d')S_0 h_0^{debased}$$

where $h_0^{debased}$ is the delta of the debased premium

$$h_0^{debased} = \frac{f_{u'} - f_{d'}}{S_0 u' - S_0 d'}$$

which is smaller than that of the accurate premium $h_0^{debased} < h_0$. Inserting $f_u - f_{u'} = (u - u')S_0 h_0^{debased}$ and $f_d - f_{d'} = (d - d')S_0 h_0^{debased}$ into the recursion for which the premium is updated, which consequently depends on the actual return factors for time decay in the option, and for which separates out the periodic mispricing from a recursion in the actual return factors and dependence for time decay in *S* (with *u* and *d*) rather than the notional asset for which the option is priced *Y* (with $u' = u/\Delta$ and $d' = d\Delta$) by the debased premium

$$\begin{aligned} f_0^{debased} &= vpf_u^{debased} + vqf_d^{debased} - \left(vp(u - u')S_0 h_0^{debased} + vq(d - d')S_0 h_0^{debased} \right) \\ &= vpf_u^{debased} + vqf_d^{debased} - \left((vpu + vqd) - vpu' - vqd' \right) S_0 h_0^{debased} \\ &= vpf_u^{debased} + vqf_d^{debased} - (vpu + vqd - 1)S_0 h_0^{debased} \end{aligned}$$

since $vpu' + vqd' = 1$ by the definition of *u*′ and *d*′. This is exactly the same form as that for which the asset is debased by varying the parameter *p* with slack. The mathematics then follows exactly the same lines. The term $(vpu + vqd - 1)S_0 h_0^{debased}$ is the periodic mispricing in the debased premium which is utilized to sink the return on the replicating market asset and keep the debased portfolio return flat at *r* for $v\Pi_1 = \Pi_0$. Similarly for the next step

$$\begin{aligned} f_u^{debased} &= vpf_{uu'}^{debased} + vqf_{ud'}^{debased} \\ &= vpf_{uu}^{debased} + vqf_{ud}^{debased} - (vpu + vqd - 1)S_u h_u^{debased} \end{aligned}$$ if the first step in *S* is to $S_0 u$ (rather than $S_0 u'$)

$$\begin{aligned} f_d^{debased} &= vpf_{du'}^{debased} + vqf_{dd'}^{debased} \\ &= vpf_{du}^{debased} + vqf_{dd}^{debased} - (vpu + vqd - 1)S_d h_d^{debased} \end{aligned}$$ if the first step in *S* is to $S_0 d$ (rather than $S_0 d'$)

and the periodic mispricing in the debased premium is then $(vpu + vqd - 1)S_u h_u^{debased}$ if the spot price of the underlying market asset moved upwards and $(vpu + vqd - 1)S_d h_d^{debased}$ if it moved downwards so that the portfolio is kept debased $v\Pi_2 = \Pi_1$ and flat at *r*. For the second step

$$f_{uu}^{debased} = vpf_{uuu}^{debased} + vqf_{uud}^{debased} - (vpu + vqd - 1)S_{uu} h_{uu}^{debased}$$ if the 2nd step in *S* is to $S_0 u^2$ (rather than $S_0 (u')^2$)

$$f_{ud}^{debased} = vpf_{udu}^{debased} + vqf_{udd}^{debased} - (vpu + vqd - 1)S_{ud} h_{ud}^{debased}$$ if the 2nd step in *S* is to $S_0 ud$ (rather than $S_0 u'd'$)

$$f_{dd}^{debased} = vpf_{ddu}^{debased} + vqf_{ddd}^{debased} - (vpu + vqd - 1)S_{dd} h_{dd}^{debased}$$ if the 2nd step in *S* is to $S_0 d^2$ (rather than $S_0 (d')^2$)

for $v\Pi_3 = \Pi_2$ and so on to maturity at the forward variables at step *n*. So from the initial recursion of the debased option premium $f_0^{debased}$, unfolding the recursion with the periodic mispricings separated out

$$
\begin{aligned}
f_0^{debased} &= vpf_u^{debased} + vqf_d^{debased} - (puv + qdv - 1)S_0 h_0^{debased} \\
&= vp\left(vpf_{uu}^{debased} + vqf_{ud}^{debased} - (puv + qdv - 1)S_u h_u^{debased}\right) + vq\left(vpf_{ud}^{debased} + vqf_{dd}^{debased} - (puv + qdv - 1)S_d h_d^{debased}\right) \\
&\qquad -(puv + qdv - 1)S_0 h_0^{debased} \\
&= v^2 p^2 f_{uu}^{debased} + 2v^2 pqf_{ud}^{debased} + v^2 q^2 f_{dd}^{debased} \\
&\qquad -vp(puv + qdv - 1)S_u h_u^{debased} - vp(puv + qdv - 1)S_d h_d^{debased} - (puv + qdv - 1)S_0 h_0^{debased} \\
&= v^2 p^2 f_{uu}^{debased} + 2v^2 pqf_{ud}^{debased} + v^2 q^2 f_{dd}^{debased} \\
&\qquad -(puv + qdv - 1)\left(vpS_u h_u^{debased} + vpS_d h_d^{debased} + S_0 h_0^{debased}\right) \\
&= v^2 p^2\left(vpf_{uuu}^{debased} + vqf_{uud}^{debased} - (puv + qdv - 1)S_{uu} h_{uu}^{debased}\right) + 2v^2 pq\left(vpf_{uud}^{debased} + vqf_{udd}^{debased} - (puv + qdv - 1)S_{ud} h_{ud}^{debased}\right) \\
&\qquad +v^2 q^2\left(vpf_{udd}^{debased} + vqf_{ddd}^{debased} - (puv + qdv - 1)S_{dd} h_{dd}^{debased}\right) \\
&\qquad -(puv + qdv - 1)\left(vpS_u h_u^{debased} + vpS_d h_d^{debased} + S_0 h_0^{debased}\right) \\
&= v^3 p^3 f_{uuu}^{debased} + 3v^3 p^2 qf_{uud}^{debased} + 3v^3 pq^2 f_{udd}^{debased} + v^3 q^3 f_{ddd}^{debased} \\
&\quad -(puv + qdv - 1)\left(v^2 p^2 S_{uu} h_{uu}^{debased} + 2v^2 pqS_{ud} h_{ud}^{debased} + v^2 q^2 S_{dd} h_{dd}^{debased} + vpS_u h_u^{debased} + vpS_d h_d^{debased} + S_0 h_0^{debased}\right) \\
&\ \ \ldots \\
&= v^n \sum_{x=0}^{n} {}^nC_x p^x q^{n-x} f_{u^x d^{n-x}}^{debased} - (puv + qdv - 1)\sum_{j=0}^{n-1} v^j \sum_{x=0}^{j} {}^jC_x p^x q^{j-x} S_{u^x d^{j-x}} h_{u^x d^{j-x}}^{debased} \\
&= f_0 - (puv + qdv - 1)\sum_{x=0}^{n-1} v^x E\left(S_x h_x^{debased} \mid S_0\right)
\end{aligned}
$$

where f_0 is the statistically accurate premium calculated with the return factors of the actual market asset due to the parity of the accurate and debased premiums for zero term to expiration at the boundary conditions from successively conditioning on *n* steps of the market asset rather than the notional demeaned asset for which the debased option is priced. Therefore, the outcome is exactly the same as that for which the option portfolio and premium are debased via the parameter for the binomial probabilities, namely the statistical bias in the debased price is that for which the returns on the traded asset are sunk by the mispricing in the debased price. In particular, the bias is the expected value of the return on assets of the replicating portfolio earned on killing the option that is passed up to subsidize the mispricing [most of which is in the initial transaction of the exchange of premium] to keep the portfolio return debased and flat at *r* and the cashflows zero and void of return

$$
\begin{aligned}
f_0 - f_0^{debased} &= Bias \\
&= (puv + qdv - 1)\sum_{j=0}^{n-1} v^j \sum_{x=0}^{j} {}^jC_x p^x q^{j-x} S_{u^x d^{j-x}} h_{u^x d^{j-x}}^{debased} \\
&= (puv + qdv - 1)\sum_{j=0}^{n-1} v^j E\left(S_j h_j^{debased} \mid S_0\right)
\end{aligned}
$$

This overlooks the numerical replicating issues due to the variation in the sample space when the return factors are changed back to source the mispricing in the premium by the replication that nullifies the option. Not ignoring these numerical issues means a portfolio that is not delta neutral, an NPV of cashflow that is not identically zero, and a subsidizing asset that does not fully source the mispricing in the debased premium earned [from the asset] by flatlining it [the option]. Evidence of this in markets is a portfolio return that is not fixed and flat at *r* and a NPV of cashflow that differs in any way from zero.

■

Summary

From the result that the trading returns on the market asset used to flatline the option are used to subsidize the mispricing in the debased premium, the following deductions can be made for a debased option portfolio and premium priced accurate only for a risky asset void of market return; when the underlying market asset earns a market rate in excess of cash, for market makers who write options and replicate with an offsetting asset to nullify and subsidize the option

> **(1) i)** The writing of a cheap (underpriced) call option at the debased price is subsidized by the excess investment returns earned on the period-to-period long asset balances in *S* traded to replicate, offset, and flatline the option;
> **(1) ii)** The writing of an expensive (overpriced) put option at the debased price subsidizes the investment losses incurred on the period-to-period short asset balances in *S* traded to replicate, offset, and flatline the option.

Return-castrated option valuation states that in order for prices to be consistent (and all debased, and trade lead for lead), all market participants must lose or pass up all exposures to all the option payoffs of all the option securities in the market, and in doing so, convert the market to one for redundant securities so that all positions in options must be nullified so that every option, each and every one for which is mispriced, can be subsidized. So additionally, from the buy side since return-castrated valuation states that all participants must replicate and nullify their options in order to misprice them, those whom are taking a long position on an option must go long at the debased price and replicate with an offsetting asset and forego the benefit of the security that they are purchasing [note that if one does not want the option, then they can save themselves the trouble and not buy it in the first place]. Consequently,

> **(2) i)** The cost saving in the purchase of a cheap (underpriced) call option at the debased price subsidizes the investment losses incurred on the period-to-period short asset balances in *S* traded to replicate, offset, and flatline the option;
> **(2) ii)** The excess premium paid in the purchase of an expensive (overpriced) put option at the debased price is subsidized by the investment returns earned on the period-to-period long asset balances in *S* traded to replicate, offset, and flatline the option.

When the underlying market asset grows at a rate less than the cash rate, the market return on the position in the underlying asset is negative for the under performing asset and there is a reverse in the direction of the trading subsidies to the option that is being flatlined. It should be [very] obvious that subsidizing flatlined options with trading returns [earned on flatlining options!] is a pointless circularity. If everyone must do this, as risk neutral valuation says, then every option is a flatlined option. In which case, it is much more efficient to shut down the market of redundant securities of debased and flatlined options, and the payoff profile of options should not be taught to students since all options are flatlined options.

Out of the context of a replicating portfolio and subsidizing the statistical error by flatlining the option, there is no way of financing the mispricing, and debased option premiums are outright wrong. How many people want to kill their option so they can sell it for half price?

It is therefore the contention of this thesis that there are no miracles to be had, particularly not by debasing option portfolios, systematically mispricing options, and denying the economic value of derivatives.

In the context of markets, every time put-call parity of debased prices is observed not to hold or implied volatility is skewed over *k*, the market is saying that prices are not debased prices;

1) observations that prices are not Black and Scholes (are not debased)
2) violations of debased put-call parity of Black and Scholes prices (violation of put-call parity of prices without a mean net of cash)
3) the volatility skew as a function of exercise price, as implied from market option prices

are three pieces of irrefutable evidence that Black and Scholes (1973) and their Black-Scholes-Merton Partial Differential Equation $rV - V_t = \frac{1}{2}\sigma^2 Y^2 V_{YY} + r\,Y\,V_Y$ are wrong and that prices are not debased and never have been, and that instead market prices explicitly recognize market return μ consistent with the Partial Differential Equation of the sensible and statistically accurate premium $rV - V_t = \frac{1}{2}\sigma^2 S^2 V_{SS} + (\mu + \frac{1}{2}\sigma^2)\,S\,V_S$ derived and argued vehemently for by this thesis; this Partial Differential Equation conceptually recognizes **derivative markets as extensions to the spot markets in *S***. This is in stark contrast to return-neutral valuation which inadvertently considers the derivative market as a redundant market for redundant securities and, further, inadvertently attempts to flatline it for an overall zero NPV which is only ever achievable by shutting the market down. Hence, risk-neutral valuation fails in this always and will only ever succeed by closing the market down in which case all cashflows will zero, all options will cease to exist, and there will be no option risk. This thesis contends that options are not redundant securities and that, instead, options have very real economic value specifically as extensions to the spot markets as deferred transactions of these markets; it is a corollary then that the essence of derivative markets derive from that of the spot markets and this essence is return μ which is otherwise debased from derivatives by the equation $rV - V_t = \frac{1}{2}\sigma^2 Y^2 V_{YY} + r\,Y\,V_Y$ and the resulting premium.

Explaining the Statistical Bias in the Debased Premiums out of the Context of a Replicating Portfolio

Whenever an option is not hedged and nullified by an offsetting asset, the debased premiums are outright wrong since there is no method of subsidizing and earning back the mispricing in the premium for the market asset castrated of return that results from debasing the option portfolio with $\Pi_0 = v\Pi_u = v\Pi_d$. If someone who buys or writes an option wants the payoff profile of the option, then they must not replicate. Then, if someone is either

not going to replicate or is not going to rebalance more than say 10,000 times (as a proxy for $n = \infty$), then the only sensible price is the premium which is statistically accurate for the market asset *S* to deliver as argued for by this thesis in the context of having proved in more ways than one that this is the accurate premium for the option on the market asset to deliver.

Instead of being barred and denied the right to earn a profit (and a living) and make the most of option markets, market participants should be encouraged to do so. This is conducive to an efficient and well functioning market for which the derivative markets are an extension and perfect complement to the spot markets. This is in direct contradiction to the debasement of derivatives and the notion that derivative markets are redundant markets of redundant securities; return is and always has been the essence of western markets.

Part I: Summary and Conclusion

In summary, the constraint $\Pi_0 = v\Pi_u = v\Pi_d$ debases the premium, the debased premium kills the return in any offsetting asset (if there is no offsetting asset then it is mispriced outright), and the replication (to subsidize the mispricing) flatlines the option turning it into a redundant security void of a payoff profile. It is consequently more efficient to close a risk neutral market down as there will then be no option payoff profiles without having to do a very large amount[24] of asset trading [with much tied up working capital] for absolutely no benefit with certainty [if it worked, which it doesn't].

The source of error in logical reasoning of papers such as those by Cox, Ross, and Rubenstein[25] and the mathematics upon which risk-neutral valuation [in a discrete setting] is based consists of; inadvertently **(1)** debasing the option portfolio by holding it flat at the cash rate and **(2)** backing out a mispriced premium, after neglecting **(i)** the economic value of the derivative and **(ii)** a changing delta which must be continuously/contiguously maintained for the replicating asset, and for which **(A)** the returns are sunk by the mispriced option [so that the market return subsidizes the mispricing] and **(B)** kills the option in doing so resulting in all cashflows that are void of market return, and **(C)** no option. This has been a misguided justification for the debasement of derivatives. This methodology denies the economic value of derivatives as deferred spot transactions and consequently infers implicitly or otherwise that derivatives are valueless, baseless, and redundant securities. The reason for this is that, prior to this thesis, it had not been proved that for an accurate premium the market asset earns the market return and the short option incurs the cash rate net of the replicating effects [which may seem rather obvious now, but it had to be proved as this thesis has done {and presented at the 2010 Finance PhD Colloquium}]. In short, they got the maths wrong by inadvertently debasing the portfolio.

Instead of spending working careers subsidizing debased premiums and never getting it right [and inadvertently just trying to return to accurate premiums with implied volatility], and an inefficient market whereby risk is always mispriced (and everyone [risk-neutral] being a failure and spending working careers trying to cover up this fact[26]), this thesis is advocating spending working careers modeling the financial variables to get accurate calculus variables to model the market variables and calculating accurate premiums accordingly, consistent with the sorts of notions from technical analysis for which debased option portfolios and premiums are at odds with, namely that **return is the essence of markets** and that **markets form dynamic systems on variables on *S*** [rather than it all just being cash with white noise]. Option replicating portfolios are definitely not for every option and should be relegated to pedagogy and special programs, and debased premiums relegated to history. From the notions of this book of this thesis, options and other derivatives in an environment that fully recognizes market return are fully complementary to technical analysis and the spot financial markets and are anything but redundant. This is a tremendous opportunity to give full recognition to the economic value of derivatives and make the most of the very exciting mathematics of BOOK 3 of this thesis and bring a proper mode of professionalism to derivatives[27] and the mathematics of derivatives, and move derivatives out of 40 years of darkness [systematically discrediting {and failing} intelligent and sensible people and regressing probability theory, among other things].

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[24] (ultra) high frequency with *n* times more transactions in the asset than in the option, and on an order of magnitude well in excess of the option itself since the level of trading in the asset always exceeds the value of the option all the way up to expiration, $S_j h_{j,n} > f_{j,n}$.

[25] Cox, J. C., Ross, S. A. and Rubinstein, M., (1979) *Option Pricing: A Simplified Approach*. Journal of Financial Economics, Vol 7, No. 3 (Sep., 1979), pp. 229-263.

[26] by blaming it on (among other things) equilibrium, volatility skew, completeness, utility, transaction costs, taxation, market inefficiencies, market frictions, probability distributions that do not exist, the risk preferences of investors, redundant securities, etc, etc, etc, etc, etc, …

[27] including an examination process and technical standards, and risk management including actuarial rate filings to regulators similarly to insurance companies which also function on principles of risk and variables on *S*.

Addendum

This is an extension to the return components for **(ii) Portfolio debased by letting the return factor *u* vary for the slack between return and no return** with additional explanation as to the effects of replicating a premium priced by *Y* by trading with *S* whereby the bias in the nominalized exercise price *a* does not adjust continuously because of being an argument in a discrete sum due to the discrete binomial basis function.

For the portfolio and premium debased by varying the parameter u subject to $u = 1 / d$ so that u' and d' are the return factors debased with slack Δ for $u' = u/\Delta$ and $d' = d\Delta$

$$u' = (1+r)\frac{1+\sqrt{1-4pqv^2}}{2p} \qquad \text{and} \qquad d' = (1+r)\frac{1-\sqrt{1-4pqv^2}}{2q}.$$

The additional terms to $f_{u'} - f_u$ and $f_{d'} - f_d$, and their effects on the portfolio returns and replication cashflow are considered now. Noting that the delta of the debased option premium is

$$h_{0,n} = v^{n-1}\sum_{x=a'}^{n-1} {}^{n-1}C_x (pu')^x (qd')^{(n-1)-x} + v^{n-1}\frac{\left(S_0(u')^a(d')^{n-a} - k\right)}{S_0(u'-d')} {}^{n-1}C_{a'-1} p^{a'-1} q^{n-a'}$$

Upsteps

If the market asset moves up, then if there is not a change in the base u'/d' characteristic of the standardized exercise price

$$\begin{aligned} a'(S_0u) &= a'(S_0\Delta u/\Delta) = a'(S_0\Delta u') \\ &= \log\left(k/S_0\Delta u'(d')^{n-1}\right)/\log(u'/d') \\ &= \log\left(k/S_0u'(d')^{n-1}\right)/\log(u'/d') - \log(\Delta)/\log(u'/d') \\ &= a'(S_0u') - \log(\Delta)/\log(u'/d') \end{aligned}$$

then the integer parts of both $a'(S_0u)$ and $a'(S_0u')$ remain equal and $\log(\Delta)/\log(u'/d')$ does not cause a decrement in the number of terms in the sum for the premium so

$$\begin{aligned} f_{u'} - f_u &= v^{n-1}\sum_{x=a'(S_0u')}^{n-1} \left(S_0(u')^{x+1}(d')^{n-1-x} - k\right) {}^{n-1}C_x p^x q^{n-1-x} - v^{n-1}\sum_{x=a'(S_0u)}^{n-1} \left(S_0u(u')^x(d')^{n-1-x} - k\right) {}^{n-1}C_x p^x q^{n-1-x} \\ &= v^{n-1}\sum_{x=a'(S_0u')}^{n-1} \left(S_0u'(u')^x(d')^{n-1-x} - S_0u(u')^x(d')^{n-1-x}\right) {}^{n-1}C_x p^x q^{n-1-x} \\ &= (u'-u)S_0\sum_{x=a'-1}^{n-1} {}^{n-1}C_x (vpu')^x (vqd')^{n-x-1} \\ &= (u'-u)S_0\sum_{x=a'}^{n-1} {}^{n-1}C_x (vpu')^x (vpd')^{n-x-1} + (u'-u)S_0\, {}^{n-1}C_{a'-1}(vpu')^{a'-1}(vqd')^{n-a'} \\ &\quad + (u'-u)S_0v^{n-1}\frac{\left(S_0(u')^a(d')^{n-a} - k\right)}{S_0(u'-d')} {}^{n-1}C_{a-1} p^{a-1} q^{n-a} - (u'-u)S_0v^{n-1}\frac{\left(S_0(u')^a(d')^{n-a} - k\right)}{S_0(u'-d')} {}^{n-1}C_{a-1} p^{a-1} q^{n-a} \\ &= (u'-u)S_0\left(\sum_{x=a'}^{n-1} {}^{n-1}C_x (vpu')^x (vqd')^{n-x-1} + v^{n-1}\frac{\left(S_0(u')^a(d')^{n-a} - k\right)}{S_0(u'-d')} {}^{n-1}C_{a-1} p^{a-1} q^{n-a}\right) \\ &\quad + (u'-u)\left(S_0(u')^{a'-1}(d')^{n-a'} - \frac{S_0(u')^{a'}(d')^{n-a'} - k}{u'-d'}\right) {}^{n-1}C_{a'-1}(vp)^{a'-1}(vq)^{n-a'} \\ &= (u'-u)S_0h_0 - (u'-u)\frac{\left(S_0(u')^{a'-1}(d')^{n-a'+1} - k\right)}{u'-d'} {}^{n-1}C_{a'-1}(vp)^{a'-1}(vq)^{n-a'} \end{aligned}$$

The standardized exercise price a' is the minimum number of upsteps to expire inthemoney, so $a'-1$ upsteps is outofthemoney and so $S_0(u')^{a'-1}(d')^{n-a'+1} - k < 0$ is negative. Also, the upstep ratio debased with slack is smaller than that of the market so $u' < u$ and $u' - u < 0$ is also negative. Therefore the term being subtracted

$$(u'-u)\frac{S_0(u')^{a'-1}(d')^{n-a'+1} - k}{u'-d'} {}^{n-1}C_{a'-1}(vp)^{a'-1}(vq)^{n-a'} = \varepsilon_u$$

is positive and hence $f_{u'} - f_u = (u'-u)S_0h_0 - \varepsilon_u$ for $\varepsilon_u > 0$.

Downstep

With $a'(S_0 d') = a'(S_0) = a'$

$$\begin{aligned} a'(S_0 d) &= a'(S_0 \Delta d/\Delta) = a'(S_0 d'/\Delta) \\ &= \log\left(k\Delta / S_0 d'(d')^{n-1}\right)/\log(u'/d') \\ &= \log\left(k / S_0 d'(d')^{n-1}\right)/\log(u'/d') + \log(\Delta)/\log(u'/d') \\ &= a'(S_0 d') + \log(\Delta)/\log(u'/d') \end{aligned}$$

If the market asset moves down, then if there is not a change in the base u'/d' characteristic of the standardized exercise price

$$a'(S_0 d) = a'(S_0 d') + \log(\Delta)/\log(u'/d')$$

then the integer parts of both $a'(S_0 d)$ and $a'(S_0 d')$ remain equal and $\log(\Delta)/\log(u'/d')$ does not cause a increment in the number of terms in the sum for the premium so

$$\begin{aligned} f_{d'} - f_d &= v^{n-1}\sum_{x=a'(S_0 d')}^{n-1} \left(S_0(u')^x(d')^{n-x} - k\right)^{n-1}C_x p^x q^{n-1-x} - v^{n-1}\sum_{x=a'(S_0 d)}^{n-1} \left(S_0 d(u')^x(d')^{n-1-x} - k\right)^{n-1}C_x p^x q^{n-1-x} \\ &= v^{n-1}\sum_{x=a'(S_0 d')}^{n-1} \left(S_0 d'(u')^x(d')^{n-1-x} - S_0 d(u')^x(d')^{n-1-x}\right)^{n-1}C_x p^x q^{n-1-x} \\ &= (d'-d)S_0 \sum_{x=a'}^{n-1} {}^{n-1}C_x (vpu')^x (vqd')^{n-x-1} \\ &= (d'-d)S_0 \left(\sum_{x=a'}^{n-1} (vu')^x (vd')^{n-x-1} {}^{n-1}C_x p^x q^{n-1-x} + v^{n-1}\frac{\left(S_0(u')^{a'}(d')^{n-a'} - k\right)}{S_0(u'-d')} {}^{n-1}C_{a'-1} p^{a'-1} q^{n-a'} \right) \\ &\qquad -(d'-d)S_0 v^{n-1}\frac{\left(S_0(u')^{a'}(d')^{n-a'} - k\right)}{S_0(u'-d')} {}^{n-1}C_{a'-1} p^{a'-1} q^{n-a'} \\ &= (d'-d)S_0 h_0 - (d'-d)S_0 \frac{\left(S_0(u')^{a'}(d')^{n-a'} - k\right)}{S_0(u'-d')} {}^{n-1}C_{a'-1} (vp)^{a'-1} (vq)^{n-a'} \end{aligned}$$

Since a' is the minimum number of upsteps to expire inthemoney, so a' upsteps is inthemoney and $S_0(u')^{a'}(d')^{n-a'} - k > 0$ is positive. Also, the downstep ratio debased with slack is larger than that of the market so $d' > d$ and $d' - d > 0$ is positive. Therefore the term being subtracted

$$(d'-d)\frac{S_0(u')^{a'}(d')^{n-a'} - k}{u'-d'} {}^{n-1}C_{a'-1}(vp)^{a'-1}(vq)^{n-a'} = \varepsilon_d$$

is positive and hence $f_{d'} - f_d = (d'-d)S_0 h_0 - \varepsilon_d$ for $\varepsilon_d > 0$.

In summary

For any step, if there is not a change in the base u'/d' characteristic of the standardized exercise price

(1) $IntegerPart\left[a'(S_0 u)\right] = IntegerPart\left[a'(S_0 u') - \log(\Delta)/\log(u'/d')\right]$ for a step upwards or

(2) $IntegerPart\left[a'(S_0 d)\right] = IntegerPart\left[a'(S_0 d') + \log(\Delta)/\log(u'/d')\right]$ for a step downwards

then

$$\begin{aligned} f_{u'} - f_u &= (u'-u)S_0 h_0 - (u'-u)\frac{\left(S_0(u')^{a'-1}(d')^{n-a'+1} - k\right)}{u'-d'} {}^{n-1}C_{a'-1}(vp)^{a'-1}(vq)^{n-a'} \\ &= (u'-u)S_0 h_0 - \varepsilon_u \\ f_{d'} - f_d &= (d'-d)S_0 h_0 - (d'-d)S_0 \frac{\left(S_0(u')^{a'}(d')^{n-a'} - k\right)}{S_0(u'-d')} {}^{n-1}C_{a'-1}(vp)^{a'-1}(vq)^{n-a'} \\ &= (d'-d)S_0 h_0 - \varepsilon_d \end{aligned}$$

so from

(i) $f_{u'} - f_u = (u' - u)S_0 h_0 - \varepsilon_u$
(ii) $f_{d'} - f_d = (d' - d)S_0 h_0 - \varepsilon_d$

which is added to the portfolio change, the change in the premium is

(i) $S_0 u h_0 - (1+r)S_0 h_0 + \varepsilon_u$ when the debased premium changes with an upstep; and
(ii) $S_0 d h_0 - (1+r)S_0 h_0 + \varepsilon_d$ when the debased premium changes with a downstep

with respect to the market asset rather than the demeaned asset for which it is priced, and the portfolio net of the rebalance outstanding is then

$\Pi_u = (1+r)S_0 h_0 - (1+r) f_u - \varepsilon_u$
$\Pi_d = (1+r)S_0 h_0 - (1+r) f_d - \varepsilon_d$

each of the numerical terms ε eat into the flat cash rate return on the market asset and the terms differ depending on whether it is an upstep or a downstep so that the portfolio return is not free from the delta and binomial basis $\Delta\Pi_j = (r - \varepsilon)\Pi_j + c\Delta S_j$. To build up a lattice of the debased premium that changes with respect to the market asset [a premium debased with slack that changes with respect to an asset which is not debased with slack] requires a separate lattice for each node of the lattice of the original variable of the market asset. Apart from the terminus with $n + 1$ nodes, a n step forward transition has a total of $1 + 2 + \ldots + n$ nodes which amounts to $\Sigma_{j=1}^{n} j = \frac{1}{2} n(n+1)$ separate lattices, which is complicated and time consuming but shows exactly what is happening with a 'return to the original variable' to fund the mispriced premium, i.e., removing the slack from the demeaned asset *Y* whilst keeping the premium and deltas debased and hoping for zero net cashflow by sourcing the mispricing in the debased price from returns earned on *S*. This makes it very clear that *S* and *Y* are necessarily separate variables.

For any step, whenever there is a change in the base u'/d' characteristic of the standardized exercise price

(1) $a'(S_0 u) = a'(S_0 u') - \log(\Delta)/\log(u'/d')$ for a step upwards or
(2) $a'(S_0 d) = a'(S_0 d') + \log(\Delta)/\log(u'/d')$ for a step downwards

then upper partial sums from $a'(S_0 u')$ have one more term in them than $a'(S_0 u) = a'(S_0 u') - \log(\Delta)/\log(u'/d')$ and upper partial sums for $a'(S_0 d')$ have one less term than $a'(S_0 d) = a'(S_0 d') + \log(\Delta)/\log(u'/d')$ which makes ε_u smaller still and ε_d larger;

(1) ε_u is reduced by a further term $v^{n-1}\left(S_0 u (u')^{a'-2}(d')^{n-a'+1} - k\right)^{n-1}C_{a'-2} p^{a'-2} q^{n-a'+1}$ and

(2) ε_d is increased by the term $v^{n-1}\left(S_0 d (u')^{a'}(d')^{n-a'-1} - k\right)^{n-1}C_{a'} p^{a'} q^{n-a'-1}$

Because it is completely irrelevant anyway [who wants to subsidize a mispriced option with trading returns earned on killing it?], I will not go on with it [and waste precious space with bulky algebra] but the point is that they do not replicate properly for movements in *S* when replicating an option priced for *Y* because of the discrete basis function and the sample spaces between *S* and *Y* that do not correspond. The outcome is a non-zero NPV from a replication that does not competently source the mispricing in the debased premium and so does not keep the portfolio debased fixed and flat at *r* for a zero NPV of cashflow. There should not be any more white noise written advocating debased premiums until someone gets bona-fide evidence of the ability to get a zero NPV of cashflow for a bona-fide option proper in bona-fide markets; and even then the debased price will still be the wrong price, and anything but a valid excuse to debased derivatives and turn them into redundant securities.

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PART II: PROPORTIONAL BROWNIAN MOTION, OPTIONS, AND REPLICATING PORTFOLIOS

This thesis refers to what is termed risk-neutral valuation as return-neutral valuation for reasons that should become apparent to any sensible reader. The debased option premium, and the PDE for this premium, denies the economic value of derivatives. Hence, return-neutral valuation, including the Black-Scholes formulae which are the debased premiums for put and call options, is nothing other than the systematic mispricing of options and is synonymous with the debasement of derivatives.

If you don't want to read the next 100 pages, just look at the debased formula and ask yourself *where is the mean μ?* Alternatively, look at the accurate formula [on page 10], and then wherever μ occurs, kill it with Δ; what you get is the debased formula. This is the wrong price for an option derivative.

Purpose and Introduction

The reason for BOOK 2 is to make the case for accurate option premiums and make possible the calculus of BOOK 3 to quantify option derivatives and present a unifying calculus for both the spot markets and the derivative markets as dynamic systems on *S*. Considered as a dynamic system, as technical analysts believe the market is, the market can then be modeled by the integral and differential calculus of stochastic basis functions on *S*. The proofs of PART II of BOOK 2 are complicated, more so than the binomial walk of PART I; tedious algebra and calculus. There is less room for discussion so the work is primarily mathematical. This is used to show correspondence with the binomial walk and all of the results regarding debased portfolios, which is the purposes of PART II, to make the case for accurate option premiums in the context of the continuous time theory as a continuation of the Raison D'etre and as the limit of the binomial results. Everything in the binomial walk limits to that in proportional Brownian motion, as the CLT takes effect and the PMF becomes a dense PDF as the step size is made small and, for a given forward transition length, the number of steps is made large in proportion to the reciprocal of the step size. Given that all functions limit to proportional Brownian motion, as the domain of attraction of the CLT for homogenous sums of random variables with finite variance, I have decided to include only a limit theorem for a random walk to continuous Brownian motion by reference to the transition MGF [and not worried about taking limits of everything else to show correspondence].

Apart from any of the qualitative discussions, the main contribution of this book is the mathematical formulae, in particular, that of the correct derivation of the PDE for the accurate option premium, the diffusion equation of the option premium and the resultant infinitesimal portfolio returns [that do not earn the cash rate but rather convex portfolio flux in market return], and the total NPV of the cashflows of the replicating portfolios [among a myriad of other original formulae]. All mathematical arguments, derivations and proofs, are from first principles.

This thesis was the first[1] to derive the correct partial differential equation of the unbiased and statistically accurate option premium, which is the entire basis of BOOK 2 of this thesis, [presented in 2010, and should have been passed outright in 2011; and since been copied] and the correct derivation of the Black-Scholes-Merton equation which is the PDE for the debased option premium priced accurately only for the demeaned asset. This derivation, rather than backing it out of portfolio returns, makes clear that the PDE for the debased premium is the wrong equation for derivatives. From the correct method of deriving the PDE of the option premium (multiplying the backward equation of the transition density by the intrinsic value, integrating and adding a source), it is shown that the correct PDE of the option premium has the same exponential drift as the market asset upon which the value of the option depends, and that which must be delivered (long or short) if the option expires inthemoney at option maturity. This is pricing the premium accurately for the transition distribution of the market asset, which is the only distribution for which the option can be priced if the premium is to be statistically accurate. The return-neutral arguments simply back out the PDE erroneously from the constituent return components of the portfolio differential resulting from their faulty premise that delta neutral portfolios must earn the cash rate. This simply debases the convex flux of the portfolio, and that of the PDE, of the term ΔhS where Δ is the debasing slack variable surreptitiously introduced as a result of holding the portfolio dynamics fixed flat at $d\Pi_t = r\Pi_t dt$. This is entirely analogous with the binomial walk and is synonymous with nothing other than the debasement of derivatives, since it implies that derivatives have no economic value. In terms of the replicating portfolio, then, it is trading returns of the market asset that then subsidize the incorrectly priced option which

[1] and if the derivation was known before this thesis [before I worked it out], return-neutral valuation would have been dispensed with long ago.

structurally misprices the option risk; a mispricing that it is only earned back by flatlining the option, and consequently implies that options under return-neutral valuation are redundant and worthless securities.

This book, then, makes an unequivocal case against the inept and hopeless return-neutral nonsense by explaining the statistical bias and inaccuracy in the debased premium and PDE of this premium, and then proposes a revision to the market philosophy upon which option markets trade risk consistent with the notion of **free** and **efficient** markets [rather than **communism** and **idiocy**], and **technically sophisticated theory**; in particular, markets for which option portfolios are free to earn return and for which prices are efficient because they are accurate for the securities that are traded within the market with statistically unbiased and accurate premiums. For technically sophisticated theory, it is proposed that the stochastic calculus of BOOK 3, as a calculus for dynamic systems on stochastic basis variables, is perfectly suited to the management and quantification of both the spot and derivative markets.

This is a call[2] for a change in methodological and theoretical positioning from pretending that risk in derivatives can be ignored, purely because risk-neutral valuation has misconstrued being "return-free" as being "risk-free" (i.e., a charlatan "risk free miracle" that says [falsely] that risk can be ignored), to that which acknowledges fully the risk of the spot and derivative markets as dynamic systems on S and strives to best understand and manage this dynamics with statistical accuracy and the stochastic calculus, and a proper mode of professionalism. It is believed, for good reason, that this will reduce the frequency and severity of derivative disasters which have plagued derivatives markets around the world [ever since return-neutral valuation became prevalent].

Definitions and Notation for Variables and Parameters

S is the market asset and is an exponential Brownian motion. The letter S implies $S(T)$ conditional on the backward variable and spot price $S(t) = S_0$; Y is the demeaned asset, and is S with a sink that demeans all of the return in excess of cash r (all the market return); V is the option premium and h is the option delta which is the partial rate of change in the option premium with respect to the underlying market asset; Δ is the slack variable that debases the market asset, the basis function, the option portfolio and the option premium, and all its derived functions such as delta; T and t (or t_0) are the forward and backward time variables respectively; statistical functions consist of $f\ x\ M\ \theta$ where f is the transition PDF, x is the forward diffusion variable and argument of the PDF, M is the transition MGF, and θ is the MGF parameter and the MGF analogue to the forward diffusion variable (i.e., variable for the transition moments rather than transition probability). B is standard Brownian motion, and d and ∂ are total and partial derivatives respectively.

Since the diffusion time dependence for PART II is continuous, rather than a discrete variable indexing steps, and the notation of the transition density function of the variable of the underlying asset dynamics is a dense PDF, the transition distribution is $f_{S(T)|S(t_0)}(x|S_0)$ in continuous rolling time as a limiting form of the PMF $p_{j,n}(x|S_0)$ of the walk in step indexing. The limit of discrete mass function p from a random walk to the continuous density function f of a Brownian motion is the DeMoivre-Laplace Limit Theorem as applicable to walks and Brownian motion.

DeMoivre-Laplace Limit Theorem – Proportional Brownian Motion

The moment generating function (MGF) of each of the scaled basis components of the random walk X_j are

$$M_{(\log u-\log d)X_j+\log d}(\theta) = pe^{\theta\log u} + qe^{\theta\log d}$$

The variable of the aggregate walk is

$$S_n = S_0 u^X d^{n-X} \qquad \text{where } X = X_1 + X_2 + \ldots + X_n \sim B(n,p)$$

The MGF of the backward variable is $M_{\log S_0}(\theta) = e^{\theta\log S_0}$. Convolving the n random variables and the backward variable for S_n by successively multiplying the MGF cumulatively gives the MGF of the logarithm of the multiplicative walk

$$M_{\log S_n|\log S_0}(\theta) = e^{\theta\log S_0}(pe^{\theta\log u} + qe^{\theta\log d})^n$$

[2] which is inthemoney and should be exercised.

which is the MGF of the returns of the binomial market asset *S* for a forward transition of *n* steps. It is now shown to limit, upon centering and scaling to zero mean unit variance, to a standard unit normal. The logarithm of the *n*-step walk variable, conditional on the spot price S_0, is

$$\begin{aligned}\log S_n &= (\log u - \log d)X + \log S_0 + n\log d \\ &\sim (\log u - \log d)B(n, p) + \log S_0 + n\log d\end{aligned}$$

The mean and variance of the logarithm of the forward *n* step transition are

$$\begin{aligned}E(\log S_n \mid \log S_0) &= (\log u - \log d)E(X) + \log S_0 + n\log d \\ &= (\log u - \log d)np + \log S_0 + n\log d \\ &= \log S_0 + n(p\log u + q\log d)\end{aligned}$$

and

$$\begin{aligned}Var(\log S_n \mid \log S_0) &= (\log u - \log d)^2 Var(X) \\ &= npq\log^2(u/d)\end{aligned}$$

both of which are composed of *n* additive terms for each of the *n* basis components for an *n*-step transition of the walk, $p\log u + q\log d$ for the basis mean and $pq\log^2(u/d)$ for the basis variance, for a total of n such additions by the n^{th} step as per linear summation of *n* cumulants for a convolution sum of *n* random variables composing the components of the summed diffusion basis function.

For continuous time arguments and the limit theorem, it is now assumed that the parameter n counting the number of steps of the forward transition to option expiration is proportional to time where each increment in time between each step is *dt* so that the discrete indexing parameter is related to the continuous chronological time variable by the relations

$$ndt = T - t_0 \qquad \text{i.e.,} \qquad n = (T - t_0)/dt$$

i.e., the total length of any forward transition corresponds to the number of steps of a random walk times the time increment between each step $T - t_0 = ndt$ and the number of steps in a random walk at any point in time equals the elapsed time since the start of the walk [the length of the forward transition] divided by the time increment between each step $n = (T - t_0)/dt$. Here, the time elapsed between each step is very small *dt*.

A binomial random walk is a unit root summation of scaled Bernoulli variables. By the Central Limit Theorem, a sum of homogenous finite mean and variance random variables tends toward a normal distribution. For a sum of Bernoulli variables, the particular form of the Central Limit Theorem is from DeMoivre and Laplace whom proved the limiting form of the binomial distribution in terms of a limit in the index *n* from the binomial PMF to the normal CDF. Therefore, this limit theorem is referred to as the DeMoivre-Laplace Limit Theorem, in the context of first order diffusion variables, specifically for random walks and Brownian motion, and is hence a proof of the Central Limit Theorem for the binomial variable of the step generator and the resulting walk variable to continuous Brownian motion. For this thesis, it is the De-Moivre Laplace limit theorem adjusted for the nuances of the proportional random walk, namely, for exponential first order diffusion The approach taken to the limit theorem is an alternative method to that by DeMoivre and Laplace (**for example, Feller[3] Vol. 2**) in that it is approached by reference to the MGF since this function shows the essence of the CLT, namely that the second order moments dominate all higher moments as the cumulants sum with *n.* The MGF of the step generator and the walk variable is expanded rather than the binomial PDF, and o(*dt*) terms in cumulants for limit arguments rather than proceeding by reference to convergence arguments of the binomial PMF to the normal CDF. The limiting MGF of the walk is then obtained by letting the number of steps become infinite ($n \to \infty$) and the time increment between each step become infinitesimally small ($dt \to 0$) for a fixed transition interval $ndt = t = T - t_0$, i.e., for the constrained limits $dt \to 0$ and $n \to \infty$ for fixed $ndt = T - t_0$. Note that all of the statistical functions of the binomial walk, including recursions of premium and deltas, limit to those of continuous proportional Brownian motion and each of which are expressions of the limit theorem. Showing one suffices for all the others and hence for the limit theorem. That in the MGF is most consistent with the notion of the Central Limit Theorem for any sum of homogenous finite variance random variables, as applicable to first order diffusion for sums rather than averages [from standard statistics for testing means, for example].

[3] Feller, W., (1966) *An Introduction to Probability Theory and Its Applications, Volume 2*. John Wiley & Sons.

In taking limits and in the context of a binomial replicating portfolio, there must be a rebalance at the end of each and every step to keep the offsetting replicating delta matched with the varying option delta. This means every market fluctuation of the binomial asset, and includes in the limit whereby the number of steps is made large to a reciprocal infinitesimal limit for a continuously varying market. This means that the number of rebalances must also be made large to the reciprocal infinitesimal limit as the step size is made small, i.e., an infinite number of steps and rebalances by making the step size a calculus limit. Rebalances then must take place infinitely often. The replicating asset and delta then requires an infinite number of rebalances if the argument is to remain valid. Any one single missed delta rebalance means that there are terms that are not fully offset from option and replicating deltas that are out of parity. Any one missed node, for example, would require a gamma neutral position and since it is impossible to neutralize gamma, gamma then enters the cashflows and cannot be removed. The theory of continuous time replicating portfolios is therefore pedagogical only since, in practice, it is not possible to rebalance infinitely often.

Letting U_n denote the standardized logarithm of the walk variable for an n step transition

$$U_n = \frac{\log S_n - \log S_0 - n(p\log u + q\log d)}{\sqrt{npq}\log(u/d)}$$

From the moment generating function of the logarithm of the walk, for an n step transition from S_0

$$M_{\log S_n}(\theta) = e^{\theta \log S_0}(pe^{\theta \log u} + qe^{\theta \log d})^n$$

the moment generating function of the standardized logarithm of the transition variable U_n is

$$\begin{aligned}
M_{U_n}(\theta) &= E\left(\exp\left(\frac{\log S_n - \log S_0 - n(p\log u + q\log d)}{\sqrt{npq}\log(u/d)}\theta\right)\right) \\
&= \exp\left(-\frac{\log S_0 + n(p\log u + q\log d)}{\sqrt{npq}\log(u/d)}\theta\right) M_{\log S_n}\left(\frac{1}{\sqrt{npq}\log(u/d)}\theta\right) \\
&= \exp\left(-\frac{n(p\log u + q\log d)}{\sqrt{npq}\log(u/d)}\theta\right)\left[p\exp\left(\frac{\log u}{\sqrt{npq}\log(u/d)}\theta\right) + q\exp\left(\frac{\log d}{\sqrt{npq}\log(u/d)}\theta\right)\right]^n \\
&= \left[p\exp\left(\frac{q\log u - q\log d}{\sqrt{npq}\log(u/d)}\theta\right) + q\exp\left(\frac{p\log d - p\log u}{\sqrt{npq}\log(u/d)}\theta\right)\right]^n \\
&= \left[p\exp\left(\frac{q}{\sqrt{npq}}\theta\right) + q\exp\left(\frac{-p}{\sqrt{npq}}\theta\right)\right]^n
\end{aligned}$$

Expanding the exponential terms within the binomial sum

$$\begin{aligned}
M_{U_n}(\theta) &= \left[p\left(1 + \frac{q}{\sqrt{npq}}\theta + \tfrac{1}{2}\frac{q^2}{npq}\theta^2 + o\left(1/n\right)\right) + q\left(1 - \frac{p}{\sqrt{npq}}\theta + \tfrac{1}{2}\frac{p^2}{npq}\theta^2 + o\left(1/n\right)\right)\right]^n \\
&= \left((p+q) + \frac{pq - pq}{\sqrt{pqn}}\theta + \tfrac{1}{2}\frac{pq^2 + qp^2}{pqn}\theta^2 + o\left(1/n\right)\right)^n \\
&= \left(1 + \tfrac{1}{2}\frac{\theta^2}{n} + o\left(1/n\right)\right)^n
\end{aligned}$$

Putting $n = (T - t_0)/dt$ and letting $dt \to 0$ so that $n \to \infty$ constrained to $ndt \to T - t_0$, which means that the number of steps n of the walk become infinitely large ($\to \infty$) while the time between each step dt become infinitesimally small ($\to 0$) for a fixed forward transition of length $T - t_0$. The transition MGF then limits to an exponential of $\frac{1}{2}\theta^2$ which is the MGF of a standard unit normal

$$M_{U_n}(\theta) \to e^{\frac{1}{2}\theta^2}$$

i.e.,

$U_n \sim N(0,1)$

For this continuous time limit, the diffusion variable of the market asset is now denoted with functional notation $S(ndt)$ for the *n* step transition, with steps of length dt, starting at time 0 and hence conditional on $S(0) = S_0$. Since $ndt = T - t_0$, the continuous diffusion variable is $S(T - t_0)$ conditional on $S(0) = S_0$, and by time homogeneity of first order diffusion, the transition variable for an forward transition of length $T - t_0$ starting at time t_0 is $S(T)$ conditional on $S(t_0) = S_{t_0}$. When the backward time variable is at t_0, rather than t, the backward diffusion variable will be denoted as S_0 rather than S_{t_0} to stress that it is a constant. When the backward variable has a subscript *t* rather than t_0, $S(t) = S_t$, it will be understood that the backward time variable is *t* rather than t_0. This is so that the backward variable can be integrated over time with respect to total derivatives. This means that PART II of this thesis uses two similar sets of notation [for the backward variables], somewhat similarly to the binomial walk of PART I.

Defining the infinitesimal parameters for an infinitesimal step size

$$\lim_{dt \to 0} \frac{p \log u + q \log d}{dt} = \mu$$

$$\lim_{dt \to 0} \frac{pq \log^2 (u/d)}{dt} = \sigma^2$$

where $\log u$ and $\log d$ are proportional to dt. In the limit, then, as the number of steps become large and the step size becomes small, the transition mean and variance become

$$\lim_{dt \to 0} n(p \log u + q \log d) = \lim_{dt \to 0} \frac{p \log u + q \log d}{dt}(T - t_0) = \mu(T - t_0)$$

$$\lim_{dt \to 0} npq \log^2 (u/d) = \lim_{dt \to 0} \frac{pq \log^2 (u/d)}{dt}(T - t_0) = \sigma^2 (T - t_0)$$

net of the backward variable. Given S_0, the limiting standardized variable U_n is unit normal

$$U_n = \frac{\log S(T) - \log S_0 - \mu(T - t_0)}{\sigma \sqrt{T - t_0}} \sim N(0,1)$$

Re-standardizing for the transition distribution of the logarithm of *S*

$$\log S(T) \sim N\left(\log S_0 + \mu(T - t_0),\ \sigma^2 (T - t_0)\right)$$

conditional on S_0 at time t_0. This is a normal variable with both parameters proportional to time and is therefore a Brownian motion on logarithms for a stochastic differential $d \log S_0 = \mu dt_0 + \sigma dB_0$. Transforming back to the original scale by exponentiating the logarithm of the transition variable gives a log-normal variable

$$S(T) \mid S(t_0) = S_0$$
$$\sim \log N\left(\log S_0 + \mu(T - t_0),\ \sigma^2 (T - t_0)\right)$$

given S_0, which is the transition distribution of a geometric Brownian motion and that from the Raison D'etre for which this transition distribution was obtained by resolving $dS_0 = (\mu + \frac{1}{2}\sigma^2) S_0 dt_0 + \sigma S_0 dB_0$ for *S*.

With the caveat that all rebalances must take place continuously, and hence infinitely often, all the results from the binomial walk and replicating portfolio carry over to continuous proportional Brownian motion. Rebalances must take place infinitely often and hence continuously, without a break, so that the CLT works and it is therefore sufficient to replicate the option delta in order to kill the option. One break in the continuous stream of instantaneous delta means that gamma and higher order terms of stochastic variation enter into the portfolio cashflows, i.e., non-negligible quadratic and higher order variation. Once gamma is present in the cashflows, it cannot be removed or offset (neutralized). This means that the position in the market asset, and every trade for rebalancing thereafter, becomes subject to full market risk within the portfolio cashflows.

The Proportional Brownian Motion Market Asset S

Statistical functions of *S*

As proved in the Raison D'etre, the log-normal transition distribution of *S*

$$S(T)\,|\,S(t_0)=S_0 \sim \log N\left(\log S_0+\mu(T-t_0),\ \sigma^2(T-t_0)\right)$$

is the statistical distribution of the resolved diffusion variable, given $S(t_0)=S_0$

$$S(T)=\exp\left(\log S_0+\mu(T-t_0)+\sigma(B_T-B_{t_0})\right)$$

for the diffusion equation

$$dS_t=\left(\mu+\tfrac{1}{2}\sigma^2\right)S_t dt+\sigma S_t dB_t$$

The transition density function *f* is

$$f_{S(T)|S(t_0)}(x\,|\,S_0)=\frac{1}{x\sigma\sqrt{T-t_0}\sqrt{2\pi}}\exp\left(-\frac{[\log x-\log S_0-\mu(T-t_0)]^2}{2\sigma^2(T-t_0)}\right)$$

and the transition cumulative density function *F* is

$$F_{S(T)|S(t_0)}(x\,|\,S_0)=F_U\left(\frac{\log x-\log S_0-\mu(T-t_0)}{\sigma(T-t_0)}\right)$$

where $F_U(\cdot)$ is the CDF of a standard unit normal $U\sim N(0,1)$. The complementary transition cumulative density function is the complement

$$\overline{F}_{S(T)|S(t_0)}(x\,|\,S_0)=1-F_{S(T)|S(t_0)}(x\,|\,S_0)=\overline{F}_U\left(\frac{\log x-\log S_0-\mu(T-t_0)}{\sigma(T-t_0)}\right)$$

where $\overline{F}_U(\cdot)$ is the complementary CDF of a standard unit normal, and the bar on *F* is consistent with the notation from set theory for complements. The transition CDF features in the accurate put option premium and the complementary transition CDF features in the accurate call option premium. The return on the market asset is the logarithm of the market asset, $\log S$. The total mean return and total variance of return for the forward transition of $T-t_0$ are the parameters of the log-normal transition distribution, net of the backward variable, $\mu(T-t_0)$ and $\sigma^2(T-t_0)$. The transition MGF of the logarithm of *S* is

$$M_{\log S(T)|\log S(t_0)}(\theta\,|\,S_0)=\exp\left(\log S_0\theta+\mu(T-t_0)\theta+\tfrac{1}{2}\sigma^2(T-t_0)\theta^2\right)$$

and the transition MGF of *S* is the infinite sum

$$\begin{aligned}M_{S(T)|S(t_0)}(\theta\,|\,S_0)&=\sum_{j=0}^{\infty}\frac{\theta^j}{j!}M_{\log S(T)|\log S(t_0)}(j\,|\,S_0)\\&=\sum_{j=0}^{\infty}\frac{\theta^j}{j!}\exp\left(j\log S_0+\mu(T-t_0)j+\tfrac{1}{2}\sigma^2(T-t_0)j^2\right)\end{aligned}$$

For a forward transition of $T-t_0$, the first two raw moments are

$$M_{\log S(T)|\log S(t_0)}(1\,|\,S_0)=\exp\left(\log S_0+\mu(T-t_0)+\tfrac{1}{2}\sigma^2(T-t_0)\right)=S_0\exp\left((\mu+\tfrac{1}{2}\sigma^2)(T-t_0)\right)$$

$$M_{\log S(T)|\log S(t_0)}(2\,|\,S_0)=\exp\left(2\log S_0+2\mu(T-t_0)+2\sigma^2(T-t_0)\right)=S_0{}^2\exp\left(2(\mu+\sigma^2)(T-t_0)\right)$$

so the transition mean of *S* is then $M_{\log S(T)|\log S(t_0)}(1\,|\,S_0)$

$$E\left(S(T)\,|\,S(t_0)=S_0\right)=S_0\exp\left((\mu+\tfrac{1}{2}\sigma^2)(T-t_0)\right)$$

and the transition variance is $M_{\log S(T)|\log S(t_0)}(2\,|\,S_0)-M_{\log S(T)|\log S(t_0)}(1\,|\,S_0)^2$

$$Var\left(S(T)\,|\,S(t_0)=S_0\right)=S_0{}^2\exp\left((2\mu+\sigma^2)(T-t_0)\right)\left(\exp\left(\sigma^2(T-t_0)\right)-1\right)$$

For a transition differential, a transition length of *dt* from t_0 to t_0+dt given that $S(t_0)=S_0$, the transition distribution is

$$dS(t_0)\,|\,S(t_0)=S_0$$
$$\sim\log N\left(\mu dt,\ \sigma^2 dt\right)$$

The mean of the stochastic differential is then

$$E\left(dS(t_0)\,|\,S_0\right)=S_0\exp\left((\mu+\tfrac{1}{2}\sigma^2)dt\right)-S_0=(\mu+\tfrac{1}{2}\sigma^2)S_0dt+o(dt)$$

and the variance is

$$Var\left(dS(t_0)\,|\,S_0\right)=S_0{}^2\exp\left((2\mu+\sigma^2)dt\right)\left(\exp\left(\sigma^2dt\right)-1\right)=S_0{}^2\sigma^2dt+o(dt)$$

These are the infinitesimal mean and variance, and infinitesimal parameters, of the first order diffusion. All higher order cumulants are o(*dt*). Therefore, the stochastic differential of proportional Brownian motion as $dt\to 0$ is normal

$$dS(t_0)\,|\,S(t_0)=S_0$$
$$\sim N\left((\mu+\tfrac{1}{2}\sigma^2)S_0dt,\ \sigma^2S_0{}^2dt\right)$$

which can be expressed with $dB_0\sim N(0,dt)$ as

$$dS_0=(\mu+\tfrac{1}{2}\sigma^2)S_0dt_0+\sigma S_0dB_0$$

conditional on $S(t_0)=S_0$. Any differential *d* or derivative d/dt of any first order diffusion variable, such as dS or dB, is by default the difference between the diffusion variable and the conditional backward variable for a transition length of *dt*. For example dS_t implies, as shorthand, $dS(t)\,|\,S(t)=S_t$ which is $S(t+dt)-S(t)$ given that (conditional on) $S(t)=S_t$ for the stochastic differential $dS_t=(\mu+\tfrac{1}{2}\sigma^2)S_tdt+\sigma S_tdB_t$.

The backwards deconvolution recurrence for the transition density *f* of *S*

The backward deconvolution integral for the transition density of proportional Brownian motion [original to this thesis] is the integral whereby the basis density function is deconvolved proportionately from the forward transition distribution at the backward variables for a fixed forward time variable

$$f_{S(T)|S(t_0)}(x\,|\,S_0)=\int_0^\infty f_X(y)\,f_{S(T)|S(t_0+dt)}(x\,|\,S_0\times y)dy$$

with log-normal basis variable $X_j\sim\log N(\mu dt,\sigma^2dt)$ which has density function

$$f_X(x)=\frac{1}{x\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\frac{(\log x-\mu)^2}{\sigma^2}}$$

Hence,

$$f_{S(T)|S(t_0)}(x\,|\,S_0)=\int_0^\infty\frac{1}{y\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\frac{(\log y-\mu)^2}{\sigma^2}}f_{S(T)|S(t_0+dt)}(x\,|\,S_0\times y)dy$$

subject to a Dirac delta density at the backward variable for zero forward transition $f_{S(t_0)|S(t_0)}(S_0\,|\,S_0)=\delta(S_0)$.

The backward equation for the transition density *f* of *S*

This is assumed knowledge from BOOK 1. For one possible derivation, see the BOOK 2 Raison D'etre. Here is another. From the backwards deconvolution recurrence of the transition density

$$f_{S(T)|S(t_0)}(x \mid S_0) = \int_0^\infty f_X(y)\, f_{S(T)|S(t_0+dt)}(x \mid S_0 \times y)dy$$

expanding $f_{S(T)|S(t_0+dt)}(x \mid S_0 \times y) = f_{S(T)|S(t_0+dt)}(x \mid e^{\log S_0 + \log y})$ in a bivariate power series around t_0 and $\log S_0$

$$f_{S(T)|S(t_0)}(x \mid S_0) = \int_0^\infty f_X(y)\left\{1 + (dt)\frac{\partial}{\partial t_0} + (\log y)\frac{\partial}{\partial \log S_0} + \tfrac{1}{2}(\log y)^2 \frac{\partial^2}{(\partial \log S_0)^2}\right\} f_{S(T)|S(t_0)}(x \mid S_0)dy + o(dt)$$

and taking the basis integral inside the expansion

$$f_{S(T)|S(t_0)}(x \mid S_0) = \left\{1 + (dt)\frac{\partial}{\partial t_0} + \left(\int_0^\infty f_X(y)(\log y)dy\right)\frac{\partial}{\partial \log S_0} + \tfrac{1}{2}\left(\int_0^\infty f_X(y)(\log y)^2 dy\right)\frac{\partial^2}{(\partial \log S_0)^2}\right\} f_{S(T)|S(t_0)}(x \mid S_0) + o(dt)$$

Since the logarithm of the basis components is normal, $\log X \sim N(\mu dt, \sigma^2 dt)$, the expectations are

(1) $\int_0^\infty f_X(y)(\log y)dy = \mu dt$

(2) $\int_0^\infty f_X(y)(\log y)^2 dy = \sigma^2 dt + (\mu dt)^2 = \sigma^2 dt + o(dt)$

so that

$$f_{S(T)|S(t_0)}(x \mid S_0) = e^{-rdt}\left\{1 + (dt)\frac{\partial}{\partial t_0} + (\mu dt)\frac{\partial}{\partial \log S_0} + \tfrac{1}{2}(\sigma^2 dt)\frac{\partial^2}{(\partial \log S_0)^2}\right\} f_{S(T)|S(t_0)}(x \mid S_0) + o(dt)$$

Subtracting $f_{S(T)|S(t_0)}(x \mid S_0)$ from both sides and dividing through by dt

$$-\frac{\partial}{\partial t_0} f_{S(T)|S(t_0)}(x \mid S_0) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{(\partial \log S_0)^2} f_{S(T)|S(t_0)}(x \mid S_0) + \mu \frac{\partial}{\partial \log S_0} f_{S(T)|S(t_0)}(x \mid S_0) + \frac{o(dt)}{dt}$$

and with

(1) $\frac{\partial}{\partial \log S_0} = \frac{\partial S_0}{\partial \log S_0}\frac{\partial}{\partial S_0} = S_0 \frac{\partial}{\partial S_0}$

(2) $\frac{\partial^2}{(\partial \log S_0)^2} = \frac{\partial S_0}{\partial \log S_0}\frac{\partial}{\partial S_0}\left(\frac{\partial S_0}{\partial \log S_0}\frac{\partial}{\partial S_0}\right) = S_0 \frac{\partial}{\partial S_0}\left(S_0 \frac{\partial}{\partial S_0}\right) = {S_0}^2 \frac{\partial^2}{\partial {S_0}^2} + S_0 \frac{\partial}{\partial S_0}$

the PDE can be written as

$$-\frac{\partial}{\partial t_0} f_{S(T)|S(t_0)}(x \mid S_0) = \tfrac{1}{2}\sigma^2 {S_0}^2 \frac{\partial^2}{\partial {S_0}^2} f_{S(T)|S(t_0)}(x \mid S_0) + \left(\mu + \tfrac{1}{2}\sigma^2\right) S_0 \frac{\partial}{\partial S_0} f_{S(T)|S(t_0)}(x \mid S_0) + \frac{o(dt)}{dt}$$

Letting $dt \to 0$ so that $o(dt)/dt \to 0$

$$-\frac{\partial}{\partial t_0} f_{S(T)|S(t_0)}(x \mid S_0) = \tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2}{\partial S_0^2} f_{S(T)|S(t_0)}(x \mid S_0) + \left(\mu + \tfrac{1}{2}\sigma^2\right) S_0 \frac{\partial}{\partial S_0} f_{S(T)|S(t_0)}(x \mid S_0)$$

which is the backward equation of the density function *f* of the transition distribution of the exponential Brownian motion of the market asset *S*. This equation describes how market return and market risk are deconvolved (removed) from the transition distribution at the lower end of the transition interval (the initial conditions) for an infinitesimal change in the backward time variable, i.e., for a shortening of the time to the forward time variable *T* by dt ; $T - (t_0 + dt) = (T - t_0) - dt$. In the context of an option, this is time decay in the deferral period of the spot transaction of the derivative.

The Statistically Unbiased Option Premia for the Market Asset S

The premium for a derivative is the **present value of the statistical expectation** of the expiry value of the derivative on the underlying market variable *S*. This is the statistically accurate price. In the context of an option, similarly to the binomial walk, this is the present value of the statistical expectation of the intrinsic value of the option when it expires. In the case of **futures**, this is the value of the deferred spot transaction in the market asset *S*. In the case of **options**, it is the value of the deferred spot transaction whenever the transaction is not turned downed due to an unfavorable value of the market asset *S* when the deferral period of the derivative expires.

OPTION PREMIUM

Call Option Premium

The payoff of a call option that expires at time *T* at exercise price *k* is the intrinsic value of the derivative when it expires

$$(S_T - k) \vee 0 = \max(S_T - k, 0)$$

This is the value of a long spot transaction deferred to time *T*. This defines the call option as a derivative. In particular, it is a long future with an option to walk away from the transaction. A call option is the option derivative equivalent to a long spot trade. If the market asset is greater than the spot price at maturity (when the deferral period expires), the transaction to buy is made and the value of the call is then worth the excess of the spot price over the exercise price $S_T - k$. Otherwise the long transaction is turned down and the value is zero. The call option premium *V* at time t_0 for a deferral of length $T - t_0$ is then the expected present value of the call option intrinsic value at expiry, which is the discounted upper limited first moment of the forward transition distribution for a forward transition interval of length $T - t_0$ from time t_0 to *T*, given that the spot price of the market asset is S_0 at time t_0

$$\begin{aligned} V(k,T;S_0,t_0) &= e^{-r(T-t_0)} E\left((S(T)-k) \vee 0 \,|\, S(t_0) = S_0\right) \\ &= e^{-r(T-t_0)} E\left(\max(S(T)-k,0) \,|\, S(t_0) = S_0\right) \end{aligned}$$

This is the value of (the premium for) the long future (which is the RHS of put-call parity) plus (a premium for) the additional optionality to turn down the future long transaction (which is the put option premium). Given that the transition density of the forward transition is $f_{S(T)|S(t_0)}(x \,|\, S_0)$, the present value of the limited statistical expectation of the payoff is

$$V(k,T;S_0,t_0) = e^{-r(T-t_0)} \int_k^\infty (x-k) f_{S(T)|S(t_0)}(x \,|\, S_0) dx$$

Note that this is a transformation of the density *f*, a transformation for which is a function of the forward diffusion variable *x* only, so that the dynamics with respect to the backward variables remains unchanged aside from the time dependence in the discount factor for present values. The call option premium then integrates to

$$\begin{aligned} V(k,T;S_0,t_0) &= e^{-r(T-t_0)} E\left(\max(S_T - k,0) \,|\, S_0\right) \\ &= e^{-r(T-t_0)} \int_k^\infty (x-k) f_{S(T)|S(0)}(x \,|\, S_0) dx \\ &= e^{-r(T-t_0)} \int_k^\infty (x-k) \frac{1}{x\sigma\sqrt{T-t_0}\sqrt{2\pi}} \exp\left(-\frac{[\log x - \log S_0 - \mu(T-t_0)]^2}{2\sigma(T-t_0)}\right) dx \\ &= S_0 e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)} \overline{F}_U\left(u_k - \sigma\sqrt{T-t_0}\right) - k e^{-r(T-t_0)} \overline{F}_U\left(u_k\right) \end{aligned}$$

where $\overline{F}_U(\cdot)$ is the **complementary cumulative density function** (complementary CDF) of a **u**nit normal variable $U \sim N(0,1)$

$$\overline{F}_U(u) = \int_u^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy$$

and u_k is the standardized value of the exercise price *k* and a normal pivot for *U*

$$u_k = \frac{\log k - \log S_0 - \mu(T - t_0)}{\sigma\sqrt{T - t_0}}$$

Put Option Premium

Similarly to a call option, the payoff of a put option that expires at time *T* at exercise price *k* is the intrinsic value of the derivative when it expires

$$(k - S_T) \vee 0 = \max(k - S_T, 0)$$

This is the value of a short spot transaction deferred to time *T*. This defines the put option as a derivative. In particular, it is a short future with an option to walk away from the transaction. A put option is the option derivative equivalent to a short spot trade. If the market asset is smaller than the spot price at maturity (when the deferral period expires), the transaction to sell is made and the value of the put is then worth the shortfall of the spot price from the exercise price $k - S_T$. Otherwise the short transaction is turned down and the value is zero. The put option premium *V* at time t_0 for a deferral of length $T - t_0$ is then the expected present value of the put option intrinsic value at expiry, which is the discounted lower limited first moment of the forward transition distribution for a forward transition interval of length $T - t_0$ from time t_0 to *T*, given that the spot price of the market asset is S_0 at time t_0

$$\begin{aligned} V(k,T;S_0,t_0) &= e^{-r(T-t_0)} E\left((k - S(T)) \vee 0 \mid S(t_0) = S_0\right) \\ &= e^{-r(T-t_0)} E\left(\max(k - S(T), 0) \mid S(t_0) = S_0\right) \end{aligned}$$

This is the value of (the premium for) the short future (which is the [negative of the] RHS of put-call parity) plus (a premium for) the additional optionality to turn down the future short transaction (which is the call option premium). Given that the transition density of the forward transition is $f_{S(T)|S(t_0)}(x \mid S_0)$, the present value of the limited statistical expectation of the payoff is

$$V(k,T;S_0,t_0) = e^{-r(T-t_0)} \int_0^k (k - x) f_{S(T)|S(t_0)}(x \mid S_0) dx$$

Note that, similarly to a call, this is a transformation of the density *f* that leaves the dynamics of the backward diffusion variable unchanged. The put option premium then integrates to

$$\begin{aligned} V(k,T;S_0,t_0) &= e^{-r(T-t_0)} E\left(\max(k - S_T, 0) \mid S_0\right) \\ &= e^{-r(T-t_0)} \int_0^k (k - x) f_{S(T)|S(0)}(x \mid S_0) dx \\ &= e^{-r(T-t_0)} \int_0^k (k - x) \frac{1}{x\sigma\sqrt{T - t_0}\sqrt{2\pi}} \exp\left(-\frac{[\log x - \log S_0 - \mu(T - t_0)]^2}{2\sigma(T - t_0)}\right) dx \\ &= k e^{-r(T-t_0)} F_U\left(u_k\right) - S_0 e^{(\mu - r + \frac{1}{2}\sigma^2)(T-t_0)} F_U\left(u_k - \sigma\sqrt{T - t_0}\right) \end{aligned}$$

where $F_U(\cdot)$ is the **cumulative density function** (CDF) of a **u**nit normal variable $U \sim N(0,1)$

$$F_U(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy$$

and, exactly for the call option premium, u_k is the standardized value of the exercise price *k* and a normal pivot for *U*

$$u_k = \frac{\log k - \log S_0 - \mu(T - t_0)}{\sigma\sqrt{T - t_0}}$$

Because a Brownian motion has an infinite domain, over 0 to ∞, there is always a non-zero probability that an inthemoney option will expire outofthemoney [and a non-zero probability that an outofthemoney option will expire inthemoney]. There are therefore no pseudo options proper in continuous time [for a continuous diffusion basis function], but there will always be options for which near/approach pseudo options., i.e., options which practically are pseudo options. The extent to which any option is a pseudo option will always be subjective to some degree.

Options proper, similarly to the binomial walk, are considered here to be options that are either atthemoney or very nearthemoney. These are mainstream options. This makes options proper consistent with the futures market which also trade at or near the money since, as it stands for current practice, the futures price *k* is set to make the futures premium zero and are hence always relatively close to being atthemoney. This then makes pseudo options and lottery options fringe options which are inefficiencies and can be exploited.

Futures Premium and Put-Call Parity

Put-call parity is the relation arising from the fact that the mean of a distribution is equal to the mean of one side of a distribution plus the mean of the other side, partitioned at and shifted by the exercise price *k*

$$c_{t_0,T} - p_{t_0,T} = E\left(S_T \mid S_0\right)v^{T-t_0} - kv^{T-t_0}$$

In terms of the integrations for expectation and hence the premiums, net of discounting, this is saying simply that

$$\int_k^\infty (x-k)\, f_S(\cdot)dx - \int_0^k (k-x)\, f_S(\cdot)dx = \int_0^\infty (x-k)\, f_S(\cdot)dx$$

For geometric Brownian motion (with $t = T - t_0$) which has a transition mean

$$E(S_T \mid S_0) = \int_0^\infty x\, f_{S(T)|S(t_0)}(x \mid S_0)dx = S_0 e^{(\mu+\frac{1}{2}\sigma^2)(T-t_0)}$$

the (undebased) market put-call parity relationship is

$$c_{t_0,T} - p_{t_0,T} = S_0 e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)} - ke^{-r(T-t_0)}$$

This is the premium *F* for a long future with exercise price (futures price) *k*, which is received by the short side of the future (see *A Consistent Framework for Derivatives in PART I*). Upon expiry of the deferral period of the future, the derivative reverts to a spot transaction. The payoff profile of a future is therefore a 45 degree line.

The accurate put option premium is alternatively obtained by put-call parity from the accurate call option premium, or visa-versa. From put-call parity, the put option premium is then

$$\begin{aligned}
p_{t_0,T} &= c_{t_0,T} - S_0 v^{T-t_0} e^{(\mu+\frac{1}{2}\sigma^2)(T-t_0)} + kv^{T-t_0} \\
&= S_0 e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}\overline{F}_U\left(u_k - \sigma\sqrt{T-t_0}\right) - ke^{-r(T-t_0)}\overline{F}_U\left(u_k\right) - S_0 e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)} + ke^{-r(T-t_0)} \\
&= S_0 e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}\left(\overline{F}_U\left(u_k - \sigma\sqrt{T-t_0}\right) - 1\right) - ke^{-r(T-t_0)}\left(\overline{F}_U\left(u_k\right) - 1\right) \\
&= ke^{-r(T-t_0)}F_U\left(u_k\right) - S_0 e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}F_U\left(u_k - \sigma\sqrt{T-t_0}\right)
\end{aligned}$$

as before.

Note that the value of the additional optionality of a call option over a long future is the premium of the call option $c_{t_0,T}$ less the premium for the long future $F_{t_0,T} = S_0 e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)} - ke^{-r(T-t_0)}$, so from put-call parity this value is

$$c_{t_0,T} - F_{t_0,T} = c_{t_0,T} - \left(S_0 e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)} - ke^{-r(T-t_0)}\right) = c_{t_0,T} - (c_{t_0,T} - p_{t_0,T}) = p_{t_0,T}$$

which is the put option premium $p_{t_0,T}$. This is the value of the option to walk away from the long spot transaction of the call if it is unfavourable to the derivative upon expiry of the deferral period of the derivative at time *T*.

Similarly, the additional optionality of a put option over a short future is the premium of the put option $p_{t_0,T}$ less the premium for the short future $ke^{-r(T-t_0)} - S_0 e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}$, so from put-call parity this value is

$$p_{t_0,T} - (-F_{t_0,T}) = p_{t_0,T} - \left(ke^{-r(T-t_0)} - S_0 e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}\right) = p_{t_0,T} - (p_{t_0,T} - c_{t_0,T}) = c_{t_0,T}$$

which is the call option premium $c_{t_0,T}$. This is the value of the option to walk away from the short spot transaction of the put if it is unfavourable to the derivative upon expiry of the deferral period of the derivative at time *T*.

Hence,

- you buy a call (rather than a long future) so that you have a put option to walk away from the long transaction
- you buy a put (rather than a short future) so that you have a call option to walk away from the short transaction

Put-call parity of options that are not debased is theory required for **matching market put-call parity**. Market put-call parity is the market price and premium of a future (and pricing philosophy) that recognizes the economic value of a derivative, and the future value of the underlying spot market. Note that undebased put-call parity returns the debased option premium to the statistically accurate premium and is a method of fitting market put-call parity to option prices. This technique of this thesis is advocated for strongly [by the author] and it is suggested that that this be the first step, in getting accurate prices consistent with the market, in the phase transition to derivative markets which are notionally not debased, and systematically mispriced, by the guiding formulae and theory, and instead strive to be accurate by fully recognizing the return and future value of the underlying spot markets from which the value of the derivative is derived [which is why they are called derivatives].

Noting that derivative prices are not debased, as notable from debased put-call parity not holding for market option prices, current practice in the futures markets is to set the futures price [the exercise price k] so that the futures premium is zero. From the [undebased] futures premium $F_{t_0,T} = c_{t_0,T} - p_{t_0,T} = S_0 e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)} - ke^{-r(T-t_0)}$, putting this to zero to set the exercise price

$$c_{t_0,T} - p_{t_0,T} = S_0 e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)} - ke^{-r(T-t_0)} = 0$$

$$ke^{-r(T-t_0)} = S_0 e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}$$

$$k = S_0 e^{(\mu+\frac{1}{2}\sigma^2)(T-t_0)}$$

which implies [implicitly, and as of this time, without recognition[4]] that the futures price [the exercise price of the future] is the mean of the forward transition of the market asset at the end of the deferral period of the derivative at the forward time variable $k = S_0 e^{(\mu+\frac{1}{2}\sigma^2)(T-t_0)}$, i.e., $k = E(S_T \mid S_{t_0})$. This gives another method of inferring future market returns from the derivative markets, either from the futures market or from the option markets with **market put-call parity**, and provides a method for comparing the futures market with the options market for consistency. This also then provides a second method of remeaning a debased option premium by reference to the derivative markets, within the context of *A Consistent Framework for Derivatives*; remeaning by reference to the futures markets or by reference to the option markets. That by reference to market put-call parity is better since it is inferred from options for the pricing of options [which is a circularity but not a pointless circularity].

Note that the mathematics of this thesis in the context of acknowledging return μ in derivative markets [which is why it has been written] provides for an almost limitless scope for empirical research in derivatives, for both futures and options. In particular, **empirical research into the crystal ball that is derivatives**.

The backwards deconvolution recurrence for the option premium *V*

Similarly to recursion for the distribution of the market asset for which the option is priced, this recurrence describes how basis components are deconvolved from the forward transition, from total return and total risk, within the formula for the option premium. This formula is derived by transforming *f* to *V*, i.e., transforming the recurrence from that of the transition density function *f* to that of the option premium *V* by transforming the dependent variable of the integral equation and discounting (adding the discount factor). This transforms the recurrence for the transition density *f* to a statistical expectation with the discount factor as required for the premium *V* as a price for the derivative. From the backwards deconvolution recurrence for the transition density *f*

$$f_{S(T)|S(t_0)}(x \mid S_0) = \int_0^\infty f_X(y) f_{S(T)|S(t_0+dt)}(x \mid S_0 \times y) dy$$

For a **call option**, multiplying both sides by the option intrinsic value $x - k$ at the forward variables

$$(x-k) f_{S(T)|S(t_0)}(x \mid S_0) = (x-k) \int_0^\infty f_X(y) f_{S(T)|S(t_0+dt)}(x \mid S_0 \times y) dy$$

and integrating over the forward diffusion variable *x* over the range of moneyness for the excess function

[4] apart from the consistent observations that **debased put-call parity** does not hold for market option prices and that hence derivative prices are not debased prices.

$$\int_k^\infty (x-k) f_{S(T)|S(t_0)}(x \mid S_0) dx = \int_k^\infty (x-k) \int_0^\infty f_X(y) f_{S(T)|S(t_0+dt)}(x \mid S_0 \times y) dy dx$$
$$= \int_0^\infty f_X(y) \left(\int_k^\infty (x-k) f_{S(T)|S(t_0+dt)}(x \mid S_0 \times y) dx \right) dy$$

Multiplying both sides by the discount factor $e^{-r(T-t_0)} = e^{-rdt} e^{-r(T-t_0-dt)}$

$$e^{-r(T-t_0)} \int_k^\infty (x-k) f_{S(T)|S(t_0)}(x \mid S_0) dx = e^{-r(T-t_0)} \int_0^\infty f_X(y) \left(\int_k^\infty (x-k) f_{S(T)|S(t_0+dt)}(x \mid S_0 \times y) dx \right) dy$$
$$= e^{-rdt} \int_0^\infty f_X(y) \left(e^{-r(T-t_0-dt)} \int_k^\infty (x-k) f_{S(T)|S(t_0+dt)}(x \mid S_0 \times y) dx \right) dy$$

The call option premium is defined as $V(k,T;S_0,t_0) = e^{-r(T-t_0)} \int_k^\infty (x-k) f_{S(T)|S(t_0)}(x \mid S_0) dx$ and hence the recurrence is whence transformed

$$V(k,T;S_0,t_0) = e^{-rdt} \int_0^\infty f_X(y) V(k,T;S_0 \times y, t_0 + dt) dy$$

where *X* is the log-normal basis variable $X_j \sim \log N(\mu dt, \sigma^2 dt)$. The recursion [original to this thesis] is subject to the call option payoff for the boundary region of the diffusion sample space of the market asset *S* where the option is inthemoney at option expiry

$$V(k,T;S(T),T) = \begin{cases} S(T) - k & \text{for } S(T) > k \\ 0 & \text{for } S(T) < k \end{cases}$$

Similarly for a **put option** whereby the intrinsic value in the forward diffusion variable is $k - x$; multiplying both sides by the option intrinsic value $k - x$ at the forward variables

$$(k-x) f_{S(T)|S(t_0)}(x \mid S_0) = (k-x) \int_0^\infty f_X(y) f_{S(T)|S(t_0+dt)}(x \mid S_0 \times y) dy$$

and integrating over the forward diffusion variable *x* over the range of moneyness for the shortfall function

$$\int_0^k (k-x) f_{S(T)|S(t_0)}(x \mid S_0) dx = \int_0^k (k-x) \int_0^\infty f_X(y) f_{S(T)|S(t_0+dt)}(x \mid S_0 \times y) dy dx$$
$$= \int_0^\infty f_X(y) \left(\int_0^k (k-x) f_{S(T)|S(t_0+dt)}(x \mid S_0 \times y) dx \right) dy$$

Multiplying both sides by the discount factor $e^{-r(T-t_0)} = e^{-rdt} e^{-r(T-t_0-dt)}$

$$e^{-r(T-t_0)} \int_0^k (k-x) f_{S(T)|S(t_0)}(x \mid S_0) dx = e^{-r(T-t_0)} \int_0^\infty f_X(y) \left(\int_0^k (k-x) f_{S(T)|S(t_0+dt)}(x \mid S_0 \times y) dx \right) dy$$
$$= e^{-rdt} \int_0^\infty f_X(y) \left(e^{-r(T-t_0-dt)} \int_0^k (k-x) f_{S(T)|S(t_0+dt)}(x \mid S_0 \times y) dx \right) dy$$

The put option premium is defined as $V(k,T;S_0,t_0) = e^{-r(T-t_0)} \int_0^k (k-x) f_{S(T)|S(t_0)}(x \mid S_0) dx$ and hence the recurrence is whence transformed

$$V(k,T;S_0,t_0) = e^{-rdt} \int_0^\infty f_X(y) V(k,T;S_0 \times y, t_0 + dt) dy$$

and is exactly the same as that for the call option. The recursion is subject to the put option payoff for the boundary region of the diffusion sample space of the market asset *S* where the option is inthemoney at option expiry

$$V(k,T;S(T),T) = \begin{cases} 0 & \text{for } S(T) > k \\ k - S(T) & \text{for } S(T) < k \end{cases}$$

This method to transform the deconvolution integral is similarly applied to the backward equation of the transition density function to derive the backward equation of the option premium, i.e., a parabolic PDE rather than a backward deconvolution integral. The backward equation is an o(*dt*) expression of the backwards (multiplicative) deconvolution integral.

Note that $V(k,T;S_0,t_0)=e^{-rdt}\int_0^{\infty} f_X(y)V(k,T;S_0\times y,t_0+dt)dy$ is the deconvolution integral of an infinite point basis variable of proportional Brownian motion and the equivalent to the (de)convolution summation $V(k,n;S_0,j)=v\sum_{y=d,u} p_X(y)V(k,n;S_0\times y,j+1)$ (i.e., $V_0=vpV_u+vqV_d$) in the binomial option premium for the two point basis variable of the proportional binomial walk.

Solving the recurrence for the option premium accurate *V* for the market asset to deliver

The backward deconvolution equation is $V(k,T;S_0,t_0)=e^{-rdt}\int_0^{\infty} f_X(y)V(k,T;S_0\times y,t_0+dt)dy$ where X is the basis variable and is distributed log-normally $X_j\sim\log N(\mu dt,\sigma^2 dt)$, noting that the integral recursion for the debased option is exactly the same but with debased basis components X', primed for slack with Δ, $X_j'\sim\log N\left(\mu' dt,\sigma^2 dt\right)\sim\log N\left((r-\frac{1}{2}\sigma^2)dt,\sigma^2 dt\right)$ for the deconvolution integral.

For the statistically accurate premium for the market asset to deliver, that which has not been debased, this is

$$V(k,T;S_0,t_0)=e^{-rdt}\int_0^{\infty}\frac{1}{\sigma y\sqrt{2\pi\,dt}}e^{-\frac{1}{2}\frac{(\log y-\mu dt)^2}{\sigma^2 dt}}V(k,T;S_0\times y,t_0+dt)dy$$

Unfolding to option expiry with *n* steps of length *dt*

$$\begin{aligned}V(k,T;S_0,t_0)&=e^{-rdt}\int_0^{\infty} f_X(y_1)V(k,T;S_0\times y_1,t_0+dt)dy_1\\&=e^{-rdt}\int_0^{\infty} f_X(y_2)e^{-rdt}\int_0^{\infty} f_X(y_1)V(k,T;S_0\times y_1\times y_2,t_0+2dt)dy_1dy_2\\&=e^{-rdt}\int_0^{\infty} f_X(y_3)e^{-rdt}\int_0^{\infty} f_X(y_2)e^{-rdt}\int_0^{\infty} f_X(y_1)V(k,T;S_0\times y_1\times y_2\times y_3,t_0+3dt)dy_1dy_2dy_3\\&\ldots\\&=e^{-rdt}\int_0^{\infty} f_X(y_n)\ldots e^{-rdt}\int_0^{\infty} f_X(y_2)e^{-rdt}\int_0^{\infty} f_X(y_1)V(k,T;S_0\times y_1\times y_2\times\ldots\times y_n,t_0+ndt)dy_1dy_2\ldots dy_n\end{aligned}$$

the option expires at $T=t_0+ndt$ in *n* periods of *dt* time units past time t_0, to which the recursion has been unfolded with *n* deconvolution integrals. At option expiration at the forward variables $T=t_0+ndt$ where $n=(T-t_0)/dt$, so **(1)** $T=t_0+ndt$ in the premium and **(2)** $nrdt=r(T-t_0)$ in the discount factor, the unfolded recursion for the option premium becomes

$$V(k,T;S_0,t_0)=e^{-r(T-t_0)}\int_0^{\infty} f_X(y_n)\ldots\int_0^{\infty} f_X(y_2)\int_0^{\infty} f_X(y_1)V(k,T;S_0\times y_1\times y_2\times\ldots\times y_n,T)dy_1dy_2\ldots dy_n$$

where, for the accurate premium, the basis density functions are

$$f_{X_j}(y_j)=\frac{1}{y_j\,\sigma\sqrt{2\pi\,dt}}e^{-\frac{1}{2}\frac{(\log y_j-\mu dt)^2}{\sigma^2 dt}}$$

which at each step deconvolves market return and market risk from the total market return and total market risk of the premium *V* for the transition interval to option expiry. Each of the scaled basis components are log-normal $X_j\sim\log N(\mu dt,\sigma^2 dt)$. The *n*-fold deconvolution of $n=(T-t_0)/dt$ basis terms from the forward transition distribution within *V* for the transition from t_0 to *T* then convolve into the *n*-fold convolution density of the *n*-step unfolded recursion

$$\begin{aligned}\textstyle\prod_1^n X_j&\sim\log N(\mu ndt,\sigma^2 ndt)\\&\sim\log N\left(\mu(T-t_0),\sigma^2(T-t_0)\right)\end{aligned}$$

Hence, the integral for the transition from the backward diffusion variable S_0 with argument $x=S_0\times y_1\times y_2\times\ldots\times y_n$ is $\prod_1^n X_j$ multiplicatively convolved onto S_0

$$S(T)=S_0\times\textstyle\prod_1^n X_j\sim\log N\left(\log S_0+\mu(T-t_0),\sigma^2(T-t_0)\right)$$

which is the forward distribution of S_T given the spot price S_0 at time t_0, and has density function

$$f_{S(T)|S(t_0)}(x\,|\,S_0)=\frac{1}{x\sigma\sqrt{ndt}\sqrt{2\pi}}e^{-\frac{1}{2}\frac{(\log x-\log S_0-\mu ndt)^2}{\sigma^2 ndt}}=\frac{1}{x\sigma\sqrt{T-t_0}\sqrt{2\pi}}e^{-\frac{1}{2}\frac{[\log x-\log S_0-\mu(T-t_0)]^2}{\sigma^2(T-t_0)}}$$

which should be familiar. The unfolded recurrence becomes

$$\begin{aligned}V(k,T;S_0,t_0)&=e^{-r(T-t_0)}\int_0^\infty f_{S_0X_1X_2\ldots X_n}(x)V(k,T;x,T)dx\\&=e^{-r(T-t_0)}\int_0^\infty f_{S(T)|S(t_0)}(x\,|\,S_0)V(k,T;x,T)dx\end{aligned}$$

i.e.,

$$V(k,T;S_0,t_0)=e^{-r(T-t_0)}\int_0^\infty \frac{1}{x\sigma\sqrt{T-t_0}\sqrt{2\pi}}e^{-\frac{1}{2}\frac{[\log x-\log S_0-\mu(T-t_0)]^2}{\sigma^2(T-t_0)}}V(k,T;x,T)dx$$

and given that the boundary is $V(k,T;S(T),T)=\max\left(S(T)-k,0\right)$ for a call option,

$$\begin{aligned}V(k,T;S_0,t_0)&=e^{-r(T-t_0)}\int_0^\infty f_{S(T)|S(t_0)}(x\,|\,S_0)V(k,T;x,T)dx\\&=e^{-r(T-t_0)}\int_0^\infty f_{S(T)|S(t_0)}(x\,|\,S_0)\max(S_T-k,0\,|\,S_T=x)dx\\&=e^{-r(T-t_0)}\int_k^\infty (x-k)\,f_{S(T)|S(t_0)}(x\,|\,S_0)dx\\&=e^{-r(T-t_0)}\int_k^\infty \frac{(x-k)}{x\sigma\sqrt{T-t_0}\sqrt{2\pi}}e^{-\frac{1}{2}\frac{[\log x-\log S_0-\mu(T-t_0)]^2}{\sigma^2(T-t_0)}}dx\\&=S_0e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}\overline{F}_U\left(u_k-\sigma\sqrt{T-t_0}\right)-ke^{-r(T-t_0)}\overline{F}_U\left(u_k\right)\end{aligned}$$

which is the definition of the premium and integrates into the formula as before for the call option premium accurate for the market asset to deliver *S*. For a put option, the boundary is $V(k,T;S(T),T)=\max\left(k-S(T),0\right)$ so the recursion solves for

$$\begin{aligned}V(k,T;S_0,t_0)&=e^{-r(T-t_0)}\int_0^\infty f_{S(T)|S(t_0)}(x\,|\,S_0)V(k,T;x,T)dx\\&=e^{-r(T-t_0)}\int_0^\infty f_{S(T)|S(t_0)}(x\,|\,S_0)\max(k-x,0\,|\,S_T=x)dx\\&=e^{-r(T-t_0)}\int_0^k (k-x)\,f_{S(T)|S(t_0)}(x\,|\,S_0)dx\\&=e^{-r(T-t_0)}\int_k^\infty \frac{(k-x)}{x\sigma\sqrt{T-t_0}\sqrt{2\pi}}e^{-\frac{1}{2}\frac{[\log x-\log S_0-\mu(T-t_0)]^2}{\sigma^2(T-t_0)}}dx\\&=ke^{-r(T-t_0)}F_U\left(u_k\right)-S_0e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}F_U\left(u_k-\sigma\sqrt{T-t_0}\right)\end{aligned}$$

which is the put option premium accurate for the market asset to deliver *S*, as required.

A. THE REPLICATING PORTFOLIO FOR THE ACCURATE OPTION PREMIUM

The Option Replicating Portfolio and Option Delta for the Market Asset S

A continuous time delta-neutral option replicating portfolio Π is established by taking an offsetting position in the underlying market asset *S*, in proportion to a quantity delta *h*, to match and offset the Gaussian basis function of the option *V* derived from the asset *S* upon which the value of the option depends

$$\Pi_{t,T}=h_{t,T}S_t-V_{t,T}$$

The total derivative and change in the replicating portfolio $d\Pi_{t,T}$ is composed of a change in the position in the market asset $d(h_{t,T}S_t)$ and a change in the value of the option premium $dV_{t,T}$

$$\frac{d\Pi_{t,T}}{dt} = \frac{d}{dt}\left(h_{t,T}S_t - V_{t,T}\right) = \frac{d(h_{t,T}S_t)}{dt} - \frac{dV_{t,T}}{dt}$$

The total change in the delta position of the asset $d(h_{t,T}S_t)$ is the change in the product of two items, the quantity delta and value of the market asset held in this quantity. Both terms in the product depend on *dS* and hence on the stochastic basis function of the diffusion variable. Since delta is dependent on both time and the stochastic basis function, similarly to $(dS_t)^2$ and peculiar to diffusion variables, the product of $dh_{t,T}$ and dS_t is not $o(dt)$. There is therefore the additional term $dh_{t,T}dS_t$ in the product rule of differentiation which means that, by one arrangement of the expression, the delta rebalance depends on the forward variable of the market asset as its current value *S* plus its change *dS*

$$\begin{aligned}\frac{d\Pi_{t,T}}{dt} &= h_{t,T}\frac{dS_t}{dt} + S_t\frac{dh_{t,T}}{dt} + \frac{dh_{t,T}dS_t}{dt} - \frac{dV_{t,T}}{dt} \\ &= h_{t,T}\frac{dS_t}{dt} + (S_t + dS_t)\frac{dh_{t,T}}{dt} - \frac{dV_{t,T}}{dt}\end{aligned}$$

The change in the asset position is then **(1)** a change in the value of the market asset $h_{t,T}dS_t$ for a fixed delta and **(2)** a change in the delta for a forward value of the market asset $(S_t + dS_t)dh_{t,T}$. The term $dh_{t,T}dS_t$ can have o(*dt*) terms separated out for a term free from *d* and forward variation in *S*. This is considered later. *S* and *V* are value variables whilst *h* is a quantity variable. Therefore, dS_t and dV_t are value changes while the term in dh_t is quantity change which requires a physical rebalance. The term $h_{t,T}dS_t$ changes within the portfolio by the value change in the market asset, as does $dV_{t,T}$, while the term $(S_t + dS_t)dh_{t,T}$ is the physical rebalance which must be paid for out of equity to complete the total portfolio change so that the replicating delta remains in parity with the instantaneous option delta. Only then does the portfolio continue to replicate and keep the delta killed [offset and neutralized]. Continuing to do so, for each and every infinitesimal *dt*, then maintains the replication so that it remains unbroken to option maturity.

It is proved [original to this thesis] that the total derivative and diffusion equation of the option premium, and analogue to the binomial total derivative, is

$$\frac{dV_{t,T}}{dt} = rV_{t,T} - \left(\mu + \tfrac{1}{2}\sigma^2\right)S_t h_{t,T} + h_{t,T}\frac{dS_t}{dt} \qquad \text{where} \quad h_{t,T} = \frac{\partial V_{t,T}}{\partial S_t}$$

so that an offsetting position in delta $h_{t,T}$ units of the market asset is sufficient to kill the Gaussian variation dS_t of the option net of the residual variation for the instantaneous delta changes. Delta is sufficient in continuous time [for a continuously varying asset and option delta] only when rebalances take place infinitely often, i.e., continuously. Taking an offsetting position in the underlying asset and replicating the option [to nullify it] with the offsetting asset then kills the option's delta and nets out the Gaussian basis function. The remaining portfolio variation is born by the delta rebalances with trading (buys and sells) in *S* to keep the replicating delta matched with the instantaneous delta of the option and keep dS_t [and dS_t^2, dS_t^3, dS_t^4,…] out of the net portfolio cashflows. This amounts to an infinite number of buys and sells in *S* for any non-infinitesimal forward transition to option expiry, to replicate and maintain a continuous variable dependent directly on the exponential Brownian motion which composes the multiplicatively aggregated basis function. This is sourced directly from equity so that the balance sheet always depends on Gaussian risk dS_t if the offsetting position in the asset is to keep the delta matched [instantaneously for all instances] to that of the option and maintain the replication. Otherwise the replication is broken and void. This thesis assumes that the replication is maintained to option maturity.

The continuous time total portfolio differential

$$\begin{aligned}d\Pi_{t,T} &= d(h_{t,T}S_t) - dV_{t,T} \\ &= h_{t,T}dS_t + S_{t+dt}dh_{t,T} - dV_{t,T} \\ &= (h_{t,T}dS_t - dV_{t,T}) + (S_t + dS_t)dh_{t,T}\end{aligned}$$

is the differential of cashflows of the replicating portfolio over the lifetime of the option being flatlined and is used to **(1)** discuss the economic value of derivatives and **(2)** explain the statistical bias and inaccuracy in the debased formulae in the context of the replicating portfolio, the **statistical bias** being the difference between the **debased price** and the **statistically accurate price**. By substituting in the diffusion equation of the option

$$dV_{t,T} = rV_{t,T}dt - \left(\mu + \tfrac{1}{2}\sigma^2\right)S_t h_{t,T}dt + h_{t,T}dS_t$$

the portfolio return components for the accurate premium, the portfolio change net of the instantaneous rebalance, are proved to be $d\Pi_{t,T} = h_{t,T}dS_t - dV_{t,T} = (\mu + \tfrac{1}{2}\sigma^2)S_t h_{t,T}dt - rV_{t,T}dt$

$$\frac{d\Pi_t}{dt} = (\mu + \tfrac{1}{2}\sigma^2)S_t \frac{\partial V_t}{\partial S_t} - rV_t$$

which is the market rate of return $\mu + \tfrac{1}{2}\sigma^2$ on the market asset *S* and the cash rate *r* on the short option *V*, and proves that the return is not fixed flat at the cash rate, unless the portfolio and premium are debased, and is instead convex flux in market return. Particularly, because of the variable composition of the two items of the portfolio, $S_t h_{t,T}$ and $V_{t,T}$ which both earn different rates, the portfolio returns are never fixed flat and are instead continually in flux and vary continuously as an interaction between **(1)** the market return $\mu + \tfrac{1}{2}\sigma^2$ earned in proportion to the continuously updated market position with respect to the instantaneous option delta and **(2)** the cash rate *r* on the path of the value of the option from time decay in the discount factor of the premium.

Critique and Limitations for Continuous Proportional [Exponential] Brownian Motion

For a theoretical replicating portfolio, in continuous time, the replicating delta does not vary unless it is rebalanced. The option delta, on the other hand, varies continuously and stochastically. The option delta is only offset if the replicating delta is continuously rebalanced as a continuous stochastic signal to exactly replicate the option delta. In PART I, this meant every binomial step of a binomial replicating portfolio, including as the number of steps became infinite for limits to continuous time. This means neutralizing every single movement in the market asset [with no exceptions]. In continuous time as considered by PART II, the option delta changes instantaneously and the matched replicating delta only remains matched for one instantaneous moment dt . One missed rebalance means that there is a break in the parity between the replicating delta and the option delta meaning that **gamma** enters into the net cashflows, and in a practical setting, almost all of the rebalances are missed so that the replicating delta is out of parity with the option delta almost all of the time. Only at a couple of instantaneous moments (of length dt) is it in parity since taking an offsetting delta position in the market asset only neutralizes the instantaneous option delta for one infinitesimal moment dt , i.e., is only delta-neutral for one instantaneous moment. The very next moment (dt), the instantaneous option delta moves out of parity with the delta position in the market asset and the position ceases to be delta-neutral. It remains out of parity, and hence is not delta neutral, until the replicating delta is next rebalanced, for which it is only delta-neutral for dt , and then for which the instantaneous option delta again instantly moves out of parity the very next infinitesimal moment and the replicating portfolio again ceases to be delta-neutral [unless it is instantaneously rebalanced again, which is impossible in actual markets]. Hence, the **replicating delta is never in parity with the instantaneous option delta** [never delta-neutral] over **any non-infinitesimal transition interval** no matter how many times it is **finitely rebalanced**.

Note that here is no correction for continuous trading. Only an infinite number of higher order diffusion terms for the higher order hedge ratios can correct for trading discretely with continuous theory. Apart from the fact that it is impossible to take an infinite number of positions for an infinite number of hedge ratios, it is not possible to take positions in even one of these higher order hedge ratios because it is not possible to invest in powers of the market asset. Increasing the rebalancing frequency to infinitesimals to replicate as a continuous signal is the only way to get continuous theory from discrete trading which is impossible, from the discreteness of transactions [a trade every second?] as well as hours of market close; evenings, weekends, and public holidays. The very fact that open-close data is recorded implies that securities vary in value over non-trading hours. The inability, then, to trade and replicate to neutralize market movements over non-trading hours of evenings, weekends, and public holidays invalidates further, and completely, any possibility of being able to replicate competently, regardless of whether you are trying to flatline the option market or earn market return after making a market in a derivative. Note that the statement that "hedging … can be made as large as we like" for a binomial walk is false and redundant. The steps are not made large, the number of steps is made large; the size of the steps are made small. Common sense will tell you that you cannot rebalance as frequently as you like, regardless of who you are or how you trade. This means that there will always be market risk from un-neutralized market fluctuations [an infinite number of them] and resulting market risk from gamma variation in net portfolio cashflow.

So continually matching the option delta in actual markets is an impossibility and the theory of continuous time replicating portfolios is theoretical only. The mathematical notions are nevertheless useful as a pedagogical tool for instruction in **the science of diffusion variables** [for which the three books of this thesis are attempting to present a complete syllabus]. In practice, the focus should be on modeling the market variable as accurately as possible and calculating a premium that is also as accurate as possible. This thesis refers to this as the *best estimate of the statistically accurate price*.

Note that replicating and offsetting an option position should be distinguished from covering a call or protecting a put. Since rebalancing infinitely often is an impossibility, successfully replicating an option and keeping the replicating asset matched with the option delta is impossible. Taking an offsetting position in the underlying market asset is, at best, a method of covering or protecting the option expiry where delta can then be interpreted as the level of confidence that the option will expire in the money, and hence **the depth of the position** in the asset **to deliver on the commitment of the derivative** at a time where it is uncertain as to whether the option will expire in or out of the money (any time before time *T*). This is not risk free and the actual payoff profile of the position is indeterminable because of immeasurable gamma (or you are all in on the position for the spot transaction for a covered call or a protected put), although the guiding principle is that you do this if you do not want the option. **The effect of the impossibility of neutralizing every market fluctuation and the resulting gamma variance in cashflows [from unmatched replicating and option deltas] is most pronounced with high variance assets [and market crashes] and least pronounced with low variance assets.** With the understanding that it is not a miracle, particularly not a risk-free miracle, and acknowledging the limitations, the theory of replicating portfolios can give guidance to what outcomes can be expected under ideals (ability to replicate infinitely often) which can then be interpolated with the realities of actual trading and market conditions, which essentially means that there will be immeasurable gamma and there will always be market risk. What this thesis is saying, then, is that taking a correct interest in the market for a replicating portfolio returns [results in] a profit for derivatives that are not debased and this is the **economic value of the derivative** as a deferred spot transaction and is what you get as earned revenue for flatlining the option position by trading with an offsetting position in the market asset to take up the pecuniary interest in the economic asset over the deferral period of the derivative (or a loss for taking a short interest in an upward moving market, for a put for example). See *A Consistent Framework for Derivatives*.

Assuming that rebalances can take place infinitely often, to neutralize every single market fluctuation and keep the instantaneous option delta matched with the replicating delta, then the position in the offsetting asset to match and kill the instantaneous option delta is in proportion to the option delta as the partial rate of change of the premium with respect to the spot price of the market asset. This is sufficient only for binomial assets and continuous Brownian motion whereby the rebalances occur infinitely often so that the CLT theorem applies and the higher order gamma terms are negligible with respect to delta.

The option delta is then $h_0 = \frac{\partial V_0}{\partial S_0}$ and the replicating portfolio is $\Pi_0 = h_0 S_0 - V_0$.

The mathematics of delta is considered now.

OPTION DELTA

The delta of the replicating asset is set to match and offset the instantaneous delta *h* of the option, which is the partial rate of change of the option premium *V* with respect to the continuously varying spot value of the underlying market asset *S*, for which the spot transaction is being deferred for the derivative

$$h(k,T;S_0,t_0) = \frac{\partial}{\partial S_0} V(k,T;S_0,t_0)$$

For a call option on a proportional [exponential] Brownian motion market variable *S*, the option delta is then

$$
\begin{aligned}
h(k,T;S_0,t_0) &= \frac{\partial}{\partial S_0}\left(S_0 e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}\overline{F}_U\left(u_k-\sigma\sqrt{T-t_0}\right)-ke^{-r(T-t_0)}\overline{F}_U\left(u_k\right)\right)\\
&= e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}\overline{F}_U\left(u_k-\sigma\sqrt{T-t_0}\right)\\
&\quad+\left(S_0 e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}\frac{\partial}{\partial S_0}\int_{u_k-\sigma\sqrt{T-t_0}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx-ke^{-r(T-t_0)}\frac{\partial}{\partial S_0}\int_{u_k}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx\right)\\
&= e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}\overline{F}_U\left(u_k-\sigma\sqrt{T-t_0}\right)\\
&\quad+\left(-S_0 e^{(\mu+\frac{1}{2}\sigma^2)(T-t_0)}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}[u_k-\sigma\sqrt{T-t_0}]^2}+k\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}[u_k]^2}\right)e^{-r(T-t_0)}\frac{\partial u_k}{\partial S_0}\\
&= e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}\overline{F}_U\left(u_k-\sigma\sqrt{T-t_0}\right)
\end{aligned}
$$

where $\overline{F}_U(\cdot)$ is the complementary CDF of a standard unit normal $U\sim N(0,1)$ and the standardized exercise u_k price is defined similarly as for the premium. For a put option, the option delta is

$$
\begin{aligned}
h(k,T;S_0,t_0) &= \frac{\partial}{\partial S_0}\left(ke^{-r(T-t_0)}F_U\left(u_k\right)-S_0 e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}F_U\left(u_k-\sigma\sqrt{T-t_0}\right)\right)\\
&= -e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}F_U\left(u_k-\sigma\sqrt{T-t_0}\right)\\
&\quad+\left(ke^{-r(T-t_0)}\frac{\partial}{\partial S_0}\int_{-\infty}^{u_k}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx-S_0 e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}\frac{\partial}{\partial S_0}\int_{-\infty}^{u_k-\sigma\sqrt{T-t_0}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx\right)\\
&= -e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}F_U\left(u_k-\sigma\sqrt{T-t_0}\right)\\
&\quad+\left(k\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}[u_k]^2}-S_0 e^{(\mu+\frac{1}{2}\sigma^2)(T-t_0)}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}[u_k-\sigma\sqrt{T-t_0}]^2}\right)e^{-r(T-t_0)}\frac{\partial u_k}{\partial S_0}\\
&= -e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}F_U\left(u_k-\sigma\sqrt{T-t_0}\right)
\end{aligned}
$$

where $F_U(\cdot)$ is the CDF of a standard unit normal $U\sim N(0,1)$. Alternatively, getting the put option delta from the call option delta and differentiated **market put-call parity**

$$
\frac{\partial}{\partial S_0}\left(c_{t_0,T}^{debased}-p_{t_0,T}^{debased}\right)=\frac{\partial}{\partial S_0}\left(S_0 e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}-ke^{-r(T-t_0)}\right)
$$

i.e.,

$$
\frac{\partial c_{t_0,T}^{debased}}{\partial S_0}-\frac{\partial p_{t_0,T}^{debased}}{\partial S_0}=e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}
$$

so

$$
\begin{aligned}
\frac{\partial p_{t_0,T}^{debased}}{\partial S_0} &= \frac{\partial c_{t_0,T}^{debased}}{\partial S_0}-e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}\\
&= e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}\overline{F}_U\left(u_k-\sigma\sqrt{T-t_0}\right)-e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}\\
&= -e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}F_U\left(u_k-\sigma\sqrt{T-t_0}\right)
\end{aligned}
$$

noting that the option deltas of the statistically accurate option premium carry the market return for **(1)** the replicating portfolio and for **(2)** the option market for which prices are not debased.

The backwards deconvolution recurrence for the option delta *h*

The backwards deconvolution recursion for the option delta [original to this thesis] is that for the option premium with the additional return factors, which bring forward the backward diffusion variable over the transition length *dt*, as a coefficient within the deconvolution integral. This is similarly to the two point binomial recursion $h_0=vpuh_u+vqdh_d$ [original to this thesis!]. One convolution or deconvolution of the continuous Gaussian basis

variable *X* over *dt* is the equivalent in continuous time to one discrete binomial step in discrete time. For an arbitrary infinitesimal at the backward variable t_0, the recursion for the option delta for the Gaussian diffusion variable is

$$\begin{aligned} h(k,T;S_0,t_0) &= \frac{\partial}{\partial S_0} V(k,T;S_0,t_0) \\ &= \frac{\partial}{\partial S_0} e^{-rdt} \int_0^\infty f_X(y) V(k,T;S_0 \times y, t_0 + dt) dy \\ &= e^{-rdt} \int_0^\infty f_X(y) \frac{\partial}{\partial S_0} V(k,T;S_0 \times y, t_0 + dt) dy \\ &= e^{-rdt} \int_0^\infty f_X(y) \frac{\partial S_0 y}{\partial S_0} \frac{\partial}{\partial S_0 y} V(k,T;S_0 \times y, t_0 + dt) dy \\ &= e^{-rdt} \int_0^\infty y \times f_X(y) h(k,T;S_0 \times y, t_0 + dt) dy \end{aligned}$$

i.e.,

$$h(k,T;S_0,t_0) = e^{-rdt} \int_0^\infty y \times f_X(y) h(k,T;S_0 \times y, t_0 + dt) dy$$

where $X_j \sim \log N(\mu dt, \sigma^2 dt)$ so

$$f_{X_j}(y) = \frac{1}{y\sigma\sqrt{2\pi\, dt}} e^{-\frac{1}{2}\frac{(\log y - \mu dt)^2}{\sigma^2 dt}}$$

and

$$h(k,T;S_0,t_0) = e^{-rdt} \int_0^\infty \frac{1}{\sigma\sqrt{2\pi\, dt}} e^{-\frac{1}{2}\frac{(\log y - \mu dt)^2}{\sigma^2 dt}} h(k,T;S_0 \times y, t_0 + dt) dy$$

Solving the recurrence for the option delta *h* accurate for the market asset to deliver

Similarly to the option premium, the backwards recursion for the option delta can be solved by unfolding it over the backward variables, in *n* infinitesimal steps of *dt*, to the forward variables at option expiration upon the end of the deferral period of the spot transaction of the derivative

$$h(k,T;S_0,t_0) = e^{-rdt} \int_0^\infty y \times f_X(y) h(k,T;S_0 \times y, t_0 + dt) dy \qquad \text{where } X_j \sim \log N(\mu dt, \sigma^2 dt)$$

Unfolding to option expiry in steps of infinitesimals *dt*

$$\begin{aligned} h(k,T;S_0,t_0) &= e^{-rdt} \int_0^\infty y_1 \times f_X(y_1) h(k,T;S_0 \times y_1, t_0 + dt) dy_1 \\ &= e^{-rdt} \int_0^\infty y_2 \times f_X(y_2) e^{-rdt} \int_0^\infty y_1 \times f_X(y_1) h(k,T;S_0 \times y_1 \times y_2, t_0 + 2dt) dy_1 dy_2 \\ &= e^{-rdt} \int_0^\infty y_3 \times f_X(y_3) e^{-rdt} \int_0^\infty y_2 \times f_X(y_2) e^{-rdt} \int_0^\infty y_1 \times f_X(y_1) h(k,T;S_0 \times y_1 \times y_2 \times y_3, t_0 + 3dt) dy_1 dy_2 dy_3 \\ &\ldots \\ &= e^{-rdt} \int_0^\infty y_n \times f_X(y_n) \ldots e^{-rdt} \int_0^\infty y_2 \times f_X(y_2) e^{-rdt} \int_0^\infty y_1 \times f_X(y_1) h(k,T;S_0 \times y_1 \times y_2 \times \ldots \times y_n, t_0 + ndt) dy_1 dy_2 \ldots dy_n \end{aligned}$$

Hence, unfolded to maturity at time *T*, the unfolded recursion for the option delta is

$$h(k,T;S_0,t_0) = e^{-nrdt} \int_0^\infty f_X(y_n) \ldots \int_0^\infty f_X(y_2) \int_0^\infty f_X(y_1) \times y_1 \times y_2 \times \ldots \times y_n \times h(k,T;S_0 \times y_1 \times y_2 \times \ldots \times y_n, t_0 + ndt) dy_1 dy_2 \ldots dy_n$$

With $n = (T - t_0)/dt$ so that $T = t_0 + ndt$, the unfolded recursion can be written as

$$h(k,T;S_0,t_0) = e^{-r(T-t_0)} \int_0^\infty f_X(y_n) \ldots \int_0^\infty f_X(y_2) \int_0^\infty f_X(y_1) \times y_1 \times y_2 \times \ldots \times y_n \times h(k,T;S_0 \times y_1 \times y_2 \times \ldots \times y_n, T) dy_1 dy_2 \ldots dy_n$$

and with $x = S_0 \times y_1 \times y_2 \times ... \times y_n$ for the statistically accurate premium and the *n*-fold deconvolution from the forward transition distribution within the delta of the recursion for time decay in the delta

$$S_0 \prod_0^n X_j dt \sim \log N(\log S_0 + \mu n dt, \sigma^2 n dt)$$
$$\sim \log N\left(\log S_0 + \mu(T - t_0), \sigma^2 (T - t_0)\right)$$

and hence an *n*-fold convolution into the aggregate density of the unfolded *n*-step recursion resulting in the forward transition distribution of the market asset *S* with density function $f_{S(T)|S(t_0)}(x \mid S_0)$. The unfolded recursion for the option delta becomes

$$h(k,T;S_0,t_0) = e^{-r(T-t_0)} \int_0^\infty f_{S_0 \times X_1 \times X_2 \times .. \times X_n}(x) \left(\frac{x}{S_0}\right) \times h(k,T;x,T) dy_1 dy_2 ... dy_n$$
$$= e^{-r(T-t_0)} \int_0^\infty f_{S(T)|S(t_0)}(x \mid S_0) \left(\frac{x}{S_0}\right) \times h(k,T;x,T) dy_1 dy_2 ... dy_n$$

where x / S_0 is the argument for the distribution of the net forward proportional transition, and is the total return on *S* over the forward transition interval to option expiry. The recursion is subject to the call option delta boundary condition, which is one long position in the market asset if the call option is inthemoney

$$h(k,T;S_T,T) = \begin{cases} 1 & if \;\; S_T > k \\ 0 & if \;\; S_T < k \end{cases}$$

The call option delta is then

$$h(k,T;S_0,t_0) = e^{-r(T-t_0)} \int_0^k \frac{x}{S_0} f_{S(t)|S(t_0)}(x \mid S_0) \times 0 \times dx + e^{-r(T-t_0)} \int_k^\infty \frac{x}{S_0} f_{S(t)|S(t_0)}(x \mid S_0) \times 1 \times dx$$
$$= e^{-r(T-t_0)} \int_k^\infty \frac{x}{S_0} f_{S(t)|S(t_0)}(x \mid S_0) dx$$

which evaluates to that as previously upon integration

$$h(k,T;S_0,t_0) = e^{(\mu + \frac{1}{2}\sigma^2 - r)(T-t_0)} \overline{F}_U\left(u_k - \sigma\sqrt{T - t_0}\right)$$

Subject to the put option delta boundary condition, which is one short position in the market asset if the put option is inthemoney

$$h(k,T;S_T,T) = \begin{cases} 0 & if \;\; S_T > k \\ -1 & if \;\; S_T < k \end{cases}$$

the put option delta from the unfolded recursion is then

$$h(k,T;S_0,t_0) = e^{-r(T-t_0)} \int_0^k \frac{x}{S_0} f_{S(T)|S(t_0)}(x \mid S_0) \times (-1) \times dx + e^{-r(T-t_0)} \int_k^\infty \frac{x}{S_0} f_{S(T)|S(t_0)}(x \mid S_0) \times 0 \times dx$$
$$= -e^{-r(T-t_0)} \int_0^k \frac{x}{S_0} f_{S(T)|S(t_0)}(x \mid S_0) dx$$

which evaluates upon integration to

$$h(k,T;S_0,t_0) = -e^{(\mu + \frac{1}{2}\sigma^2 - r)(T-t_0)} F_U\left(u_k - \sigma\sqrt{T - t_0}\right)$$

OPTION GAMMA

The backwards deconvolution recurrence for the option gamma

Differentiating the recursion for the option delta results in the recursion for the option gamma [original to this thesis]

$$\begin{aligned}\Gamma(k,T;S_0,t_0) &= \frac{\partial^2}{\partial S_0^{\,2}} V(k,T;S_0,t_0) \\ &= \frac{\partial}{\partial S_0} h(k,T;S_0,t_0) \\ &= e^{-rdt}\int_0^\infty y \times f_X(y) \frac{\partial}{\partial S_0} h(k,T;S_0 \times y, t_0+dt)dy \\ &= e^{-rdt}\int_0^\infty y \times f_X(y) \frac{\partial S_0 y}{\partial S_0} \frac{\partial}{\partial S_0 y} h(k,T;S_0 \times y, t_0+dt)dy \\ &= e^{-rdt}\int_0^\infty y^2 \times f_X(y) \Gamma(k,T;S_0 \times y, t_0+dt)dy\end{aligned}$$

i.e.,

$$\Gamma(k,T;S_0,t_0) = e^{-rdt}\int_0^\infty y^2 \times f_X(y) \Gamma(k,T;S_0 \times y, t_0+dt)dy$$

where the basis distribution is log-normal $X_j \sim \log N(\mu dt, \sigma^2 dt)$

$$f_{X_j}(y) = \frac{1}{y\sigma\sqrt{2\pi\, dt}} e^{-\frac{1}{2}\frac{(\log y - \mu dt)^2}{\sigma^2 dt}}$$

This backwards recursion is the infinite point Gaussian basis variable equivalent to the two point binomial recursion for gamma $\Gamma_0 = vpu^2\Gamma_u + vqd^2\Gamma_d$ [original to this thesis].

Table 1: Backwards Deconvolution Integrals for Recursion in *f*, *V*, Δ, Γ

Replicating Portfolio Functions	Backwards Deconvolution Recursion
***f*, density**	$f_{S(T)\|S(t_0)}(x \mid S_0) = \int_0^\infty f_X(y) f_{S(T)\|S(t_0+dt)}(x \mid S_0 \times y)dy$
***V*, option premium**	$V(k,T;S_0,t_0) = e^{-rdt}\int_0^\infty f_X(y) V(k,T;S_0 \times y, t_0+dt)dy$
***h*, option delta**	$h(k,T;S_0,t_0) = e^{-rdt}\int_0^\infty y \times f_X(y) h(k,T;S_0 \times y, t_0+dt)dy$
Γ, option gamma	$\Gamma(k,T;S_0,t_0) = e^{-rdt}\int_0^\infty y^2 \times f_X(y) \Gamma(k,T;S_0 \times y, t_0+dt)dy$

The Backward Equation of the Statistically Accurate Option Premium

The backward equation of the statistically accurate option premium $rV - V_t = \frac{1}{2}\sigma^2 S^2 V_{SS} + (\mu + \frac{1}{2}\sigma^2)SV_S$ (which can then be debased if so desired, by removing $(\mu + \frac{1}{2}\sigma^2 - r)SV_S$ from the flux) is obtained by transforming the dependent variable of the backward equation of the transition density function *f* of the market asset *S* to that of the option premium *V*, the present value of the statistical expectation of the option payoff. The mathematical transformation of $f_{S(T)\,|\,S(t_0)}(x \mid S_0)$ is then defined as

$$V_0 = V(k,T;S_0,t_0) = \begin{cases} e^{-r(T-t_0)}\int_k^\infty (x-k) f_{S(T)|S(t_0)}(x \mid S_0)dx \\ e^{-r(T-t_0)}\int_0^k (k-x) f_{S(T)|S(t_0)}(x \mid S_0)dx \end{cases} = \begin{cases} S_0 e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}\overline{F}_Y\left(y_k - \sigma\sqrt{T-t_0}\right) - ke^{-r(T-t_0)}\overline{F}_Y\left(y_k\right) \\ ke^{-r(T-t_0)}F_Y\left(y_k\right) - S_0 e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}F_Y\left(y_k - \sigma\sqrt{T-t_0}\right) \end{cases}$$

for a transition from time t_0 at which time the underlying market value has spot value S_0 to when the option expires at time *T*. This is a forward transition and term-to-maturity of $T - t_0$ time units from time t_0. The premium is then a transformation that is a function of the forward diffusion variable *x* only whereas the derivatives of the backward equation are with respect to the backward variables t_0 and S_0. Therefore, the backward equation of the option premium *V* only differs from the dynamics of the transition probability density function *f* in that it has one additional term for a source for time decay in the discount factor.

This equation describes the process of successive infinitesimal deconvolutions of basis components from within the option premium *V* from the beginning of the cumulative sum of the diffusion that comprises the forward transition to option expiry. This amounts to the deconvolution of successive components of market return and

market risk from the total market return and total market risk that constitutes the forward transition [since it is Gaussian] to option expiration, net of cash time value.

This thesis has derived four methods of obtaining the PDE of the statistically accurate option premium [original to this thesis]

(1) Limit of the backward recurrence of the premium for the binomial walk
(2) Limit of the backward [integral] recurrence of the premium for Brownian motion
(3) First principles from MGF of additive Brownian motion [MGF of log *S*]
(4) First Principles from MGF of exponential Brownian motion [MGF of *S*]

and **(5)** it can also be backed out from guessed portfolio return components. The first principles approach from the MGF of additive diffusion is contained in the BOOK 2 Raison D'etre. The limit of the recursion for the premium of the walk, which is omitted, is essentially a limit theorem as a result of the DeMoivre-Laplace Limit Theorem for the binomial PMF. Contained here are the derivations for **(2)** the limit of the recurrence, **(3 & 4)** the transforming the backward PDE of the transition density function (which is first principles but starting at the backward equation [now assumed knowledge]), and **(5)** backing it out from guessed portfolio return components. Note that backing out the PDE of the premium from guessed portfolio returns is not a rigorous method of deriving the PDE. It is included here only because this is how return-neutral valuation derives it [by this method and this method only], and in doing so, has inadvertently debased derivatives.

Derivation 1: OPTION PDE as an o(*dt*) expression of the backward deconvolution integral

The one difference between this and that of the backward equation of the density, aside from the dependent variable in *V* rather than *f*, is that this includes the discount factor. From the backwards convolution recurrence of the option premium

$$V(k,T;S_0,t_0)=e^{-rdt}\int_0^\infty f_X(y)V(k,T;S_0\times y,t_0+dt)dy$$

expanding $V(k,T;S_0\times y,t_0+dt)=V(k,T;e^{\log S_0+\log y},t_0+dt)$ in a bivariate power series around t_0 and $\log S_0$

$$V(k,T;S_0,t_0)=e^{-rdt}\int_0^\infty f_X(y)\left\{1+(dt)\frac{\partial}{\partial t_0}+(\log y)\frac{\partial}{\partial\log S_0}+\tfrac{1}{2}(\log y)^2\frac{\partial^2}{(\partial\log S_0)^2}\right\}V(k,T;S_0,t_0)dy+o(dt)$$

and taking the basis integral inside the expansion

$$V(k,T;S_0,t_0)$$
$$=e^{-rdt}\left\{1+(dt)\frac{\partial}{\partial t_0}+\left(\int_0^\infty f_X(y)(\log y)dy\right)\frac{\partial}{\partial\log S_0}+\tfrac{1}{2}\left(\int_0^\infty f_X(y)(\log y)^2dy\right)\frac{\partial^2}{(\partial\log S_0)^2}\right\}V(k,T;S_0,t_0)+o(dt)$$

with $\log X\sim N(\mu dt,\sigma^2dt)$, the expectations in the logarithm of the basis component are

(1) $\int_0^\infty f_X(y)(\log y)dy=\mu dt$

(2) $\int_0^\infty f_X(y)(\log y)^2dy=\sigma^2dt+(\mu dt)^2=\sigma^2dt+o(dt)$

so that

$$V(k,T;S_0,t_0)=e^{-rdt}\left\{1+(dt)\frac{\partial}{\partial t_0}+(\mu dt)\frac{\partial}{\partial\log S_0}+\tfrac{1}{2}(\sigma^2dt)\frac{\partial^2}{(\partial\log S_0)^2}\right\}V(k,T;S_0,t_0)+o(dt)$$

and expanding the discount factor e^{-rdt}

$$V(k,T;S_0,t_0)=\left\{1+(dt)\frac{\partial}{\partial t_0}+(\mu dt)\frac{\partial}{\partial\log S_0}+\tfrac{1}{2}(\sigma^2dt)\frac{\partial^2}{(\partial\log S_0)^2}\right\}V(k,T;S_0,t_0)-(rdt)V(k,T;S_0,t_0)+o(dt)$$

with a whole heap of *o*(*dt*) terms. Subtracting $V(k,T;S_0,t_0)$ from both sides and dividing through by *dt*, the expression can then written as

$$rV(k,T;S_0,t_0)-\frac{\partial}{\partial t_0}V(k,T;S_0,t_0)=\tfrac{1}{2}\sigma^2\frac{\partial^2}{(\partial\log S_0)^2}V(k,T;S_0,t_0)+\mu\frac{\partial}{\partial\log S_0}V(k,T;S_0,t_0)+\frac{o(dt)}{dt}$$

With

(1) $\dfrac{\partial}{\partial \log S_0} = \dfrac{\partial S_0}{\partial \log S_0}\dfrac{\partial}{\partial S_0} = S_0\dfrac{\partial}{\partial S_0}$

(2) $\dfrac{\partial^2}{(\partial \log S_0)^2} = \dfrac{\partial S_0}{\partial \log S_0}\dfrac{\partial}{\partial S_0}\left(\dfrac{\partial S_0}{\partial \log S_0}\dfrac{\partial}{\partial S_0}\right) = S_0\dfrac{\partial}{\partial S_0}\left(S_0\dfrac{\partial}{\partial S_0}\right) = {S_0}^2\dfrac{\partial^2}{\partial {S_0}^2} + S_0\dfrac{\partial}{\partial S_0}$

the PDE is arranged into the form

$$rV(k,T;S_0,t_0) - \frac{\partial}{\partial t_0}V(k,T;S_0,t_0) = \tfrac{1}{2}\sigma^2{S_0}^2\frac{\partial^2}{\partial {S_0}^2}V(k,T;S_0,t_0) + \left(\mu + \tfrac{1}{2}\sigma^2\right)S_0\frac{\partial}{\partial S_0}V(k,T;S_0,t_0) + \frac{o(dt)}{dt}$$

Letting $dt \to 0$ so that $o(dt)/dt \to 0$

$$rV(k,T;S_0,t_0) - \frac{\partial}{\partial t_0}V(k,T;S_0,t_0) = \tfrac{1}{2}\sigma^2{S_0}^2\frac{\partial^2}{\partial {S_0}^2}V(k,T;S_0,t_0) + \left(\mu + \tfrac{1}{2}\sigma^2\right)S_0\frac{\partial}{\partial S_0}V(k,T;S_0,t_0)$$

which is the PDE of the statistically accurate option premium $V_0 = V(k,T;S_0,t_0)$

$$rV_0 - \frac{\partial V_0}{\partial t_0} = \tfrac{1}{2}\sigma^2 S_0^2\frac{\partial^2 V_0}{\partial S_0^2} + \left(\mu + \tfrac{1}{2}\sigma^2\right)S_0\frac{\partial V_0}{\partial S_0}$$

subject to the boundary conditions

$$V(k,T;S(T),T) = \begin{cases} S(T)-k & if\ \ S(T) > k \\ 0 & if\ \ S(T) < k \end{cases} \qquad \text{and} \qquad V(k,T;S(T),T) = \begin{cases} 0 & if\ \ S(T) > k \\ k - S(T) & if\ \ S(T) < k \end{cases}$$

for a call option and a put option respectively.

Derivation 2: OPTION PDE as a transformed backward equation

This is contained in the Raison D'etre and is repeated here with more detail as well as the steps for that of a put option. The derivation of the PDE of the statistically accurate premium [and the replicating portfolio returns] is the essence of BOOK 2 of this thesis since this is the PDE which corrects the PDE which has been the justification over the last 40 years to debase and systematically misprice derivatives (and discredit intelligent and sensible people).

The PDE of the option premia will now be derived as a transformation of the dependent variable, the transition density function, of the backward equation.

The transition density gives the probability of a movement from a value of S_0 at time t_0 to a distribution of values x at time T. The variables S_0 and t_0 are the backward variables while x and T are the forward variables. The backward variables are variables of a known outcome, a realization of the distribution $S(t_0) = S_0$, while the forward variables are variables of a distribution $S(T)$ conditional on the backward variables. The forward diffusion variable is the argument of the transition density function. The backward equation of the proportional Brownian motion, that which is to be transformed, is

$$-\frac{\partial}{\partial t_0}f_{S(T)|S(t_0)}(x\,|\,S_0) = \tfrac{1}{2}\sigma^2 S_0^2\frac{\partial^2}{\partial S_0^2}f_{S(T)|S(t_0)}(x\,|\,S_0) + (\mu + \tfrac{1}{2}\sigma^2)S_0\frac{\partial}{\partial S_0}f_{S(T)|S(t_0)}(x\,|\,S_0)$$

which is a PDE in the backward variables (the independent variables), that is, a PDE relating the differential changes in the transition density function to the differential changes in the two independent backward variables. The dependent variable is the transition density of the diffusion variable, for a transition from the value S_0 at time t_0 to some value x at time T at option expiration, which is the density function of a log-normal distribution with transition mean $\log S_0 + \mu(T - t_0)$ and transition variance $\sigma^2(T - t_0)$

$$f_{S(T)|S(t_0)}(x\,|\,S_0) = \frac{1}{x\sigma\sqrt{T-t_0}\sqrt{2\pi}}\exp\left(-\frac{[\log x - \log S_0 - \mu(T-t_0)]^2}{2\sigma^2(T-t_0)}\right) \qquad T\forall[t_0,\infty) \text{ and } x\forall[0,\infty)$$

The option premia are the integrals of this function weighted by the expiry intrinsic value, a weighting function that is dependent on the forward diffusion variable only. The dependent variable of the backward equation, the transition density *f*, is to be transformed to the option premium *V* where *V* is defined as

$V(k,T;S_0,t_0) = e^{-r(T-t_0)} \int_k^\infty (x-k) f_{S(T)|S(t_0)}(x|S_0) dx$ for a call option premium; and

$V(k,T;S_0,t_0) = e^{-r(T-t_0)} \int_0^k (k-x) f_{S(T)|S(t_0)}(x|S_0) dx$ for a put option premium

which are discounted expected values with respect to the forward transition of the market asset *S*.

PDE for the unbiased call option premium for the market asset S

Considering a call option. For a call option premium, multiplying the backward equation in the transition PDF by the intrinsic value of the option at expiry $x-k$, which a function of the forward diffusion variable, and integrating from k upwards over the region where the call option is in the money (and the deferred long transaction is not turned down) with respect to the forward diffusion variable at option expiry at time T

$$-\int_k^\infty (x-k) \frac{\partial}{\partial t_0} f_{S(T)|S(t_0)}(x|S_0) dx = \tfrac{1}{2}\sigma^2 S_0^2 \int_k^\infty (x-k) \frac{\partial^2}{\partial S_0^2} f_{S(T)|S(t_0)}(x|S_0) dx + (\mu + \tfrac{1}{2}\sigma^2) S_0 \int_k^\infty (x-k) \frac{\partial}{\partial S_0} f_{S(T)|S(t_0)}(x|S_0) dx$$

This is a PDE in the backward variables so the integrals with respect to the forward variables can be taken inside the backward derivatives

$$-\frac{\partial}{\partial t_0} \int_k^\infty (x-k) f_{S(T)|S(t_0)}(x|S_0) dx = \tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2}{\partial S_0^2} \int_k^\infty (x-k) f_{S(T)|S(t_0)}(x|S_0) dx + (\mu + \tfrac{1}{2}\sigma^2) S_0 \frac{\partial}{\partial S_0} \int_k^\infty (x-k) f_{S(T)|S(t_0)}(x|S_0) dx$$

Multiplying both sides of the equation by the discount factor $e^{-r(T-t_0)}$, an exponential function in both the forward and backward time variables,

$$-e^{-r(T-t_0)} \frac{\partial}{\partial t_0} \int_k^\infty (x-k) f_{S(T)|S(t_0)}(x|S_0) dx$$

$$= e^{-r(T-t_0)} \tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2}{\partial S_0^2} \int_k^\infty (x-k) f_{S(T)|S(t_0)}(x|S_0) dx + e^{-r(T-t_0)} (\mu + \tfrac{1}{2}\sigma^2) S_0 \frac{\partial}{\partial S_0} \int_k^\infty (x-k) f_{S(T)|S(t_0)}(x|S_0) dx$$

Recalling the product rule $Df(t_0)v(t_0) = f(t_0)Dv(t_0) + v(t_0)Df(t_0)$, so by the converse of the product rule, using $v(t_0)Df(t_0) = Df(t_0)v(t_0) - f(t_0)Dv(t_0)$

$$e^{-r(T-t_0)} \frac{\partial f}{\partial t_0} = \frac{\partial (e^{-r(T-t_0)} f)}{\partial t_0} - r e^{-r(T-t_0)} f$$

to transform the time side of the equation. Also taking the exponential function directly inside the derivatives on the diffusion side for

$$re^{-r(T-t_0)} \int_k^\infty (x-k) f_{S(T)|S(t_0)}(x|S_0) - \frac{\partial}{\partial t_0} \left(e^{-r(T-t_0)} \int_k^\infty (x-k) f_{S(T)|S(t_0)}(x|S_0) \right) dx$$

$$= \tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2}{\partial S_0^2} \left(e^{-r(T-t_0)} \int_k^\infty (x-k) f_{S(T)|S(t_0)}(x|S_0) dx \right) + e^{-r(T-t_0)} (\mu + \tfrac{1}{2}\sigma^2) S_0 \frac{\partial}{\partial S_0} \left(e^{-r(T-t_0)} \int_k^\infty (x-k) f_{S(T)|S(t_0)}(x|S_0) \right) dx$$

which is the transformed equation in the statistically accurate and unbiased call option premium V

$$r\, V(k,T;S_0,t_0) - \frac{\partial}{\partial t_0} V(k,T;S_0,t_0) = \tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2}{\partial S_0^2} V(k,T;S_0,t_0) + (\mu + \tfrac{1}{2}\sigma^2) S_0 \frac{\partial}{\partial S_0} V(k,T;S_0,t_0)$$

subject to the boundary condition

$$V(k,T;S(T),T) = \begin{cases} S(T) - k & \text{if } S(T) > k \\ 0 & \text{if } S(T) < k \end{cases}$$

at time T, at which point the backward variable meets the forward variable and the derivative reverts to a spot transaction (if it is not turned down). In simplified notation, with subscripts denoting partial derivatives, this is $0 = \tfrac{1}{2}\sigma^2 S^2 V_{SS} + (\mu + \tfrac{1}{2}\sigma^2) S V_S + V_t - rV$. This equation is a conjugate parabolic heat equation with proportional

diffusion derivatives (the Euler-Cauchy equation in the diffusion derivatives) which is the conjugate heat equation on logarithms with convection (drift) and lateral heat loss (at rate *r*). This equation is the same form as the Black-Scholes-Merton PDE for the debased price of a derivative security with the simple difference of the correct underlying rate of return, $\mu + \frac{1}{2}\sigma^2$. The PDE with the correct flux is hence the equation for the price which has not been debased with slack and is the statistically accurate and unbiased price for the market asset to deliver *S* underlying the derivative. Note that this equation applies for constant volatility σ only.

PDE for the unbiased put option premium for the market asset S

Considering a put option. The form of the PDE of the put option premium is identical to that of the call option. The definition of the dependent variable differs in that it is the discounted expected value of the shortfall function rather than the excess function. This is expressed in the boundary conditions for which the PDE is solved for the initial value.

The transformation from the transition density to the put option premium, like that for the call option, is a function of the forward diffusion variable only. Taking the backward equation for the forward transition density and multiplying by $k - x$ and integrating over all parts of the domain for which the shortfall function is non-zero

$$-\int_0^k (k-x)\frac{\partial}{\partial t_0} f_{S(T)|S(t_0)}(x\,|\,S_0)dx = \tfrac{1}{2}\sigma^2 S_0^2 \int_0^k (k-x)\frac{\partial^2}{\partial S_0^2} f_{S(T)|S(t_0)}(x\,|\,S_0)dx + (\mu+\tfrac{1}{2}\sigma^2)S_0\int_0^k (k-x)\frac{\partial}{\partial S_0} f_{S(T)|S(t_0)}(x\,|\,S_0)dx$$

The weighting factor and integrals are in the forward variables and the derivatives in the PDE are in the backward variables. Taking the integrals inside the backward derivatives

$$-\frac{\partial}{\partial t_0}\int_0^k (k-x) f_{S(T)|S(t_0)}(x\,|\,S_0)dx = \tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2}{\partial S_0^2}\int_0^k (k-x) f_{S(T)|S(t_0)}(x\,|\,S_0)dx + (\mu+\tfrac{1}{2}\sigma^2)S_0\frac{\partial}{\partial S_0}\int_0^k (k-x) f_{S(T)|S(t_0)}(x\,|\,S_0)dx$$

Multiplying both sides of the equation by the exponential discount factor, $e^{-r(T-t_0)}$, a function of both forward and backward time variables,

$$\begin{aligned}&-e^{-r(T-t_0)}\frac{\partial}{\partial t_0}\int_0^k (k-x) f_{S(T)|S(t_0)}(x\,|\,S_0)dx \\ &\quad = e^{-r(T-t_0)}\tfrac{1}{2}\sigma^2 S_0^2\frac{\partial^2}{\partial S_0^2}\int_0^k (k-x) f_{S(T)|S(t_0)}(x\,|\,S_0)dx + e^{-r(T-t_0)}(\mu+\tfrac{1}{2}\sigma^2)S_0\frac{\partial}{\partial S_0}\int_0^k (k-x) f_{S(T)|S(t_0)}(x\,|\,S_0)dx\end{aligned}$$

and taking the discount factor inside the derivatives on **(1)** the diffusion side of the equation directly and **(2)** that of the time side using the converse of the product differentiation rule $v(t_0)Df(t_0) = Df(t_0)v(t_0) - f(t_0)Dv(t_0)$

$$\begin{aligned}&re^{-r(T-t_0)}\int_0^k (k-x) f_{S(T)|S(t_0)}(x\,|\,S_0)dx - \frac{\partial}{\partial t_0}\left(e^{-r(T-t_0)}\int_0^k (k-x) f_{S(T)|S(t_0)}(x\,|\,S_0)dx\right) \\ &\quad = \tfrac{1}{2}\sigma^2 S_0^2\frac{\partial^2}{\partial S_0^2}\left(e^{-r(T-t_0)}\int_0^k (k-x) f_{S(T)|S(t_0)}(x\,|\,S_0)dx\right) + (\mu+\tfrac{1}{2}\sigma^2)S_0\frac{\partial}{\partial S_0}\left(e^{-r(T-t_0)}\int_0^k (k-x) f_{S(T)|S(t_0)}(x\,|\,S_0)dx\right)\end{aligned}$$

which is equivalent to adding a source *rV* to the backward dynamics to remove an infinitesimal component from the discount function as the transition interval is infinitesimally shorted by increasing the backward time variable by *dt* [from $T - t_0$ to $(T - t_0) - dt$]. This equation is then the PDE in terms of the transformed dependent variable for the put option premium

$$rV(k,T;S_0,t_0) - \frac{\partial}{\partial t_0}V(k,T;S_0,t_0) = \tfrac{1}{2}\sigma^2 S_0^2\frac{\partial^2}{\partial S_0^2}V(k,T;S_0,t_0) + (\mu+\tfrac{1}{2}\sigma^2)S_0\frac{\partial}{\partial S_0}V(k,T;S_0,t_0)$$

which is $0 = \frac{1}{2}\sigma^2 S^2 V_{SS} + (\mu+\frac{1}{2}\sigma^2)SV_S + V_t - rV$ with partial derivatives as subscripts. This equation is identical to that of the call option but is subject to the complementary boundary region for the exercise value, that from the market put-call parity relationship at option expiry

$$V(k,T;S(T),T) = \begin{cases} 0 & if\ \ S(T) > k \\ k - S(T) & if\ \ S(T) < k \end{cases}$$

Derivation 3: OPTION PDE as a limit of the walk recurrence for the premium (omitted)

Derivation 4: OPTION PDE, backing out the backwards recurrence for the statistically accurate premium from a delta neutral option replicating portfolio

It can be surmised (guessed) that the replicating portfolio composed of an offsetting delta position in the underlying market asset and a short position in the option

$$\Pi_0 = S_0 h_0 - V_0 \qquad \text{where} \qquad h_0 = \frac{\partial V_0}{\partial S_0}$$

earns the market rate of return on the market asset and incurs the cash rate on the option. This is proved by numerous methods shortly. With surmising this (knowing that it is true, otherwise guessing), then expanding the option premium in a series and equating it to the portfolio return components then backs out the backward PDE of the option which is ordinarily an input to the portfolio return components. The backed out PDE has the correct rate of exponential drift and is hence accurate for the option on the market asset to deliver.

Differentiating the portfolio $\Pi_0 = S_0 h_0 - V_0$ for

$$\frac{d\Pi_0}{dt_0} = (S_0 + dS_0)\frac{dh_0}{dt_0} + h_0 \frac{dS_0}{dt_0} - \frac{dV_0}{dt_0}$$

Ignoring the delta update this is

$$\frac{d\Pi_0}{dt_0} = h_0 \frac{dS_0}{dt_0} - \frac{dV_0}{dt_0}$$

(1) Surmising that, net of the rebalance due, the portfolio earns the market return on market asset and cash rate on option

$$\frac{d\Pi_0}{dt_0} = (\mu + \tfrac{1}{2}\sigma^2) S_0 h_0 - r V_0 \qquad \text{for} \qquad h_0 = \frac{\partial V_0}{\partial S_0}$$

(2) Alternatively, expanding the option premium $V_0 = V(S_0, t_0)$ in a power series

$$\begin{aligned}\frac{dV_0}{dt_0} &= \frac{\partial V_0}{\partial t_0}\frac{dt_0}{dt_0} + \frac{\partial V_0}{\partial S_0}\frac{dS_0}{dt_0} + \tfrac{1}{2}\frac{\partial^2 V_0}{\partial S_0^2}\frac{(dS_0)^2}{dt_0} \\ &= \frac{\partial V_0}{\partial t_0} + \frac{\partial V_0}{\partial S_0}\frac{dS_0}{dt_0} + \tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 V_0}{\partial S_0^2}\end{aligned}$$

Substituting this into the portfolio derivative and cancelling the option delta with the replicating delta

$$\frac{d\Pi_0}{dt_0} = h_0 \frac{dS_0}{dt_0} - \left(\frac{\partial V_0}{\partial t_0} + \frac{\partial V_0}{\partial S_0}\frac{dS_0}{dt_0} + \tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 V_0}{\partial S_0^2} \right) = -\frac{\partial V_0}{\partial t_0} - \tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 V_0}{\partial S_0^2}$$

(1) and **(2)** now provide two expressions for an identity in the portfolio differential $\frac{d\Pi_0}{dt_0} \equiv \frac{d\Pi_0}{dt_0}$

$$\frac{d\Pi_0}{dt_0} = (\mu + \tfrac{1}{2}\sigma^2) S_0 \frac{\partial V_0}{\partial S_0} - r V_0 \qquad \text{and} \qquad \frac{d\Pi_0}{dt_0} = -\frac{\partial V_0}{\partial t_0} - \tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 V_0}{\partial S_0^2}$$

Equating the two equivalents for the identity for the portfolio change then backs out the backward equation of the premium, which is an input to the portfolio returns (rather than guessing the returns), with the correct return components

$$(\mu + \tfrac{1}{2}\sigma^2) S_0 \frac{\partial V_0}{\partial S_0} - r V_0 = -\frac{\partial V_0}{\partial t_0} - \tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 V_0}{\partial S_0^2}$$

i.e.,

$$rV_0 - \frac{\partial V_0}{\partial t_0} = \tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 V_0}{\partial S_0^2} + (\mu + \tfrac{1}{2}\sigma^2) S_0 \frac{\partial V_0}{\partial S_0}$$

which is the PDE of the statistically unbiased and accurate option premium. If the portfolio is debased with the constraint that the return is fixed flat at cash $d\Pi_0 = r\Pi_0 dt_0$ and consequently has no market return despite trading the market asset, then the debased PDE for the debased premium is backed out of the identity, that which has been debased of the term $(\mu + \frac{1}{2}\sigma^2 - r)S_0 h_0$ from the flux. This constraint $d\Pi_0 = r\Pi_0 dt_0$ surreptitiously debases the portfolio and the option premium with the slack variable $\Delta = \mu + \frac{1}{2}\sigma^2 - r$ that kills the market return and all benefit of trading an economic market asset [to take up the pecuniary interest in the economic asset for the deferred spot transaction]. This constraint debases derivatives and implies that derivatives have no economic value and are, not only mispriced securities, but also redundant securities since they have to be flatlined in order to be subsidized. If an option is not flatlined by taking an offsetting position in a replicating asset, there is no means by which to earn back the mispricing in the debased premium. The correct equation for derivatives is that with the correct rate of exponential drift at the market rate of return μ of the market asset *S* as per the equation $rV - V_t = \frac{1}{2}\sigma^2 S^2 V_{SS} + (\mu + \frac{1}{2}\sigma^2) S V_S$ derived and original to this thesis. This equation gives full recognition to return μ in derivatives as deferred transactions of the spot markets.

Backward Equation of Put-Call Parity and the Futures Premium

The backward equation of put-call parity and the futures premium are free from gamma and are hence free from the Gaussian basis function. The futures premium is free from normal CDFs, as is the delta of the futures premium, because the gamma of the put and call option premiums in the put-call parity expression cancel. Taking the backward equation of a call option and a put option

$$rC_0 - \frac{\partial C_0}{\partial t_0} = \tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 C_0}{\partial S_0^2} + (\mu + \tfrac{1}{2}\sigma^2) S_0 \frac{\partial C_0}{\partial S_0} \qquad \text{and} \qquad rP_0 - \frac{\partial P_0}{\partial t_0} = \tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 P_0}{\partial S_0^2} + (\mu + \tfrac{1}{2}\sigma^2) S_0 \frac{\partial P_0}{\partial S_0}$$

and subtracting the two for the backward equation of the futures premium (a 45 degree payoff profile)

$$r(C_0 - P_0) - \frac{\partial}{\partial t_0}(C_0 - P_0) = \tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2}{\partial S_0^2}(C_0 - P_0) + (\mu + \tfrac{1}{2}\sigma^2) S_0 \frac{\partial}{\partial S_0}(C_0 - P_0)$$

$$r(S_0 e^{(\mu+\frac{1}{2}\sigma^2 - r)(T-t_0)} - ke^{-r(T-t_0)}) - \frac{\partial}{\partial t_0}(S_0 e^{(\mu+\frac{1}{2}\sigma^2 - r)(T-t_0)} - ke^{-r(T-t_0)})$$

$$= \tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2}{\partial S_0^2}(S_0 e^{(\mu+\frac{1}{2}\sigma^2 - r)(T-t_0)} - ke^{-r(T-t_0)}) + (\mu + \tfrac{1}{2}\sigma^2) S_0 \frac{\partial}{\partial S_0}(S_0 e^{(\mu+\frac{1}{2}\sigma^2 - r)(T-t_0)} - ke^{-r(T-t_0)})$$

and since $\frac{1}{2}\sigma^2 S_0^2 \frac{\partial^2}{\partial S_0^2}(S_0 e^{(\mu+\frac{1}{2}\sigma^2 - r)(T-t_0)} - ke^{-r(T-t_0)}) = 0$

$$r(S_0 e^{(\mu+\frac{1}{2}\sigma^2 - r)(T-t_0)} - ke^{-r(T-t_0)}) - \frac{\partial}{\partial t_0}(S_0 e^{(\mu+\frac{1}{2}\sigma^2 - r)(T-t_0)} - ke^{-r(T-t_0)}) = (\mu + \tfrac{1}{2}\sigma^2) S_0 \frac{\partial}{\partial S_0}(S_0 e^{(\mu+\frac{1}{2}\sigma^2 - r)(T-t_0)} - ke^{-r(T-t_0)})$$

which is the PDE of market put-call parity and an undebased futures premium $F_0 = S_0 e^{(\mu+\frac{1}{2}\sigma^2 - r)(T-t_0)} - ke^{-r(T-t_0)}$ paid to the short side by the long side for a deferred spot transaction at price *k* (as laybuy of a financial security of the spot markets)

$$\frac{\partial F_0}{\partial t_0} = rF_0 - (\mu + \tfrac{1}{2}\sigma^2) S_0 \frac{\partial F_0}{\partial S_0}$$

which is free from gamma and hence market variation, i.e., that for *F* is exactly the backward equation of the option *V* without the gamma. This equation allocates earned premium for the financial service provided by the derivative (as a deferred transaction) from the premium received for the future as the deferral period of the derivative shortens for a forward movement in the backward variable t_0. The delta of the futures premium is

$$\frac{\partial F_0}{\partial S_0} = e^{(\mu+\frac{1}{2}\sigma^2 - r)(T-t_0)}$$

which is free from normal distribution functions and hence independent of the forward diffusion variable and the Gaussian basis function *dB*. Why? because there is no uncertainty that the spot transaction will be made at the end of the deferral period of the derivative (and hence the 45 degree line for the payoff) and hence does not depend on forward probability. It depends, instead, only on **forward market return** as the economic value of the underlying spot market. The diffusion equation of delta for a future is therefore not stochastic and gamma dependent whereas that for an option is.

The diffusion equation of then delta gives some idea of the statistical variation of replicating delta with a continually updated position in the market asset. For a future it is deterministic while for an option it is not. This is a reminder that the infinite number of rebalances for an option each depend on gamma risk. Only if all of the infinite number of rebalances are made [which is impossible] is gamma kept out of the portfolio cashflows for a continuous time option replicating portfolio.

Further PDEs of Option Related Statistical Functions

Further parabolic PDEs of option related statistical functions are now developed, those for the option delta and the option gamma. These are then utilized to derive the diffusion equations of the option delta and gamma, similarly as the option premium. These are all original to this thesis, similarly to the binomial results; this thesis should be passed without a single doubt in the world.

OPTION DELTA

The option delta was defined as the partial derivative of the option premium with respect to the spot price and backward variable of the underlying market asset

$$h(k,T;S_0,t_0)=\frac{\partial}{\partial S_0}V(k,T;S_0,t_0) \qquad\qquad h_0=\frac{\partial V_0}{\partial S_0}$$

From the backward equation of the transition PDF

$$rV_0-\frac{\partial V_0}{\partial t_0}=\tfrac{1}{2}\sigma^2S_0^2\frac{\partial^2V_0}{\partial S_0^2}+\left(\mu+\tfrac{1}{2}\sigma^2\right)S_0\frac{\partial V_0}{\partial S_0}$$

differentiating partially with respect to the backward diffusion variable

$$\frac{\partial}{\partial S_0}\left(rV_0-\frac{\partial V_0}{\partial t_0}\right)=\frac{\partial}{\partial S_0}\left(\tfrac{1}{2}\sigma^2S_0^2\frac{\partial^2V_0}{\partial S_0^2}+\left(\mu+\tfrac{1}{2}\sigma^2\right)S_0\frac{\partial V_0}{\partial S_0}\right)$$

$$\left(r\frac{\partial V_0}{\partial S_0}-\frac{\partial}{\partial t_0}\frac{\partial V_0}{\partial S_0}\right)=\tfrac{1}{2}\sigma^2\frac{\partial}{\partial S_0}\left(S_0^2\frac{\partial}{\partial S_0}\frac{\partial V_0}{\partial S_0}\right)+\left(\mu+\tfrac{1}{2}\sigma^2\right)\frac{\partial}{\partial S_0}\left(S_0\frac{\partial V_0}{\partial S_0}\right)$$

which is expressed in terms of the option delta *h* rather than the option premium *V* as

$$rh_0-\frac{\partial h_0}{\partial t_0}=\tfrac{1}{2}\sigma^2\frac{\partial}{\partial S_0}\left(S_0^2\frac{\partial h_0}{\partial S_0}\right)+\left(\mu+\tfrac{1}{2}\sigma^2\right)\frac{\partial}{\partial S_0}(S_0h_0)$$

$$=\tfrac{1}{2}\sigma^2\left(S_0^2\frac{\partial^2h_0}{\partial S_0^2}+2S_0\frac{\partial h_0}{\partial S_0}\right)+\left(\mu+\tfrac{1}{2}\sigma^2\right)\left(S_0\frac{\partial h_0}{\partial S_0}+h_0\right)$$

i.e.,

$$-\frac{\partial h_0}{\partial t_0}=\tfrac{1}{2}\sigma^2S_0^2\frac{\partial^2h_0}{\partial S_0^2}+\left(\mu+\tfrac{3}{2}\sigma^2\right)S_0\frac{\partial h_0}{\partial S_0}+\left(\mu+\tfrac{1}{2}\sigma^2-r\right)h_0 \qquad\qquad h_0=\frac{\partial V_0}{\partial S_0}$$

which is the PDE of the option delta *h* for the statistically accurate option premium. Alternatively, utilizing the same technique as the option premium *V* ; integrating the backward equation for the transition density to transform the dependent variable to that of the cumulative or complementary cumulative transition density, which is then the evaluated form of the undiscounted option delta, and then adding a source for time decay in the discount factor. For a call option this is

$$-e^{-r(T-t_0)}\int_{y_k-\sigma\sqrt{T-t_0}}^{\infty}\left(\frac{\partial}{\partial t_0}f_{S(T)|S(t_0)}(x\,|\,S_0)\right)dx$$
$$=e^{-r(T-t_0)}\int_{y_k-\sigma\sqrt{T-t_0}}^{\infty}\left(\tfrac{1}{2}\sigma^2S_0^2\frac{\partial^2}{\partial S_0^2}f_{S(T)|S(t_0)}(x\,|\,S_0)+\left(\mu+\tfrac{1}{2}\sigma^2\right)S_0\frac{\partial}{\partial S_0}f_{S(T)|S(t_0)}(x\,|\,S_0)\right)dx$$

which then transforms to

$$-\frac{\partial}{\partial t_0}h(k,T;S_0,t_0)=\tfrac{1}{2}\sigma^2S_0^2\frac{\partial^2}{\partial S_0^2}h(k,T;S_0,t_0)+\left(\mu+\tfrac{3}{2}\sigma^2\right)S_0\frac{\partial}{\partial S_0}h(k,T;S_0,t_0)+\left(\mu+\tfrac{1}{2}\sigma^2-r\right)h(k,T;S_0,t_0)$$

This is the limiting form of the binomial recursion $h_0=vpuh_u+vqdh_d$ for the step size made infinitesimally small in proportion to $dt\to 0$.

OPTION GAMMA

The option gamma is the second order partial derivative of the option premium with respect to the spot price and backward diffusion variable of *S*

$$\Gamma(k,T;S_0,t_0)=\frac{\partial^2}{\partial S_0{}^2}V(k,T;S_0,t_0)$$

This is also the first order partial derivative of the option delta with respect to the backward diffusion variable

$$\Gamma(k,T;S_0,t_0)=\frac{\partial}{\partial S_0}h(k,T;S_0,t_0)$$

So from the backward equation of the option delta

$$-\frac{\partial h_0}{\partial t_0}=\tfrac{1}{2}\sigma^2S_0^2\frac{\partial^2h_0}{\partial S_0^2}+\left(\mu+\tfrac{3}{2}\sigma^2\right)S_0\frac{\partial h_0}{\partial S_0}+\left(\mu+\tfrac{1}{2}\sigma^2-r\right)h_0$$

differentiating partially with respect to the backward diffusion variable

$$-\frac{\partial}{\partial S_0}\left(\frac{\partial h_0}{\partial t_0}\right)=\frac{\partial}{\partial S_0}\left(\tfrac{1}{2}\sigma^2S_0^2\frac{\partial^2h_0}{\partial S_0^2}+\left(\mu+\tfrac{3}{2}\sigma^2\right)S_0\frac{\partial h_0}{\partial S_0}+\left(\mu+\tfrac{1}{2}\sigma^2-r\right)h_0\right)$$
$$-\left(\frac{\partial}{\partial t_0}\frac{\partial h_0}{\partial S_0}\right)=\tfrac{1}{2}\sigma^2\frac{\partial}{\partial S_0}\left(S_0^2\frac{\partial}{\partial S_0}\frac{\partial h_0}{\partial S_0}\right)+\left(\mu+\tfrac{3}{2}\sigma^2\right)\frac{\partial}{\partial S_0}\left(S_0\frac{\partial h_0}{\partial S_0}\right)+\left(\mu+\tfrac{1}{2}\sigma^2-r\right)\frac{\partial h_0}{\partial S_0}$$

which is expressed in terms of the option gamma Γ rather than the option delta *h* as

$$-\frac{\partial\Gamma_0}{\partial t_0}=\tfrac{1}{2}\sigma^2\frac{\partial}{\partial S_0}\left(S_0^2\frac{\partial\Gamma_0}{\partial S_0}\right)+\left(\mu+\tfrac{3}{2}\sigma^2\right)\frac{\partial}{\partial S_0}\left(S_0\Gamma_0\right)+\left(\mu+\tfrac{1}{2}\sigma^2-r\right)\Gamma_0$$
$$=\tfrac{1}{2}\sigma^2\left(S_0^2\frac{\partial^2\Gamma_0}{\partial S_0^2}+2S_0\frac{\partial\Gamma_0}{\partial S_0}\right)+\left(\mu+\tfrac{3}{2}\sigma^2\right)\left(S_0\frac{\partial\Gamma_0}{\partial S_0}+\Gamma_0\right)+\left(\mu+\tfrac{1}{2}\sigma^2-r\right)\Gamma_0$$

i.e.,

$$-\frac{\partial\Gamma_0}{\partial t_0}=\tfrac{1}{2}\sigma^2S_0^2\frac{\partial^2\Gamma_0}{\partial S_0^2}+\left(\mu+\tfrac{5}{2}\sigma^2\right)S_0\frac{\partial\Gamma_0}{\partial S_0}+\left(2\mu+2\sigma^2-r\right)\Gamma_0 \qquad \Gamma_0=\frac{\partial^2V_0}{\partial S_0^2}$$

which is the PDE of the option gamma Γ for the statistically accurate option premium. This is the limiting form of the binomial recursion $\Gamma_0=vpu^2\Gamma_u+vqd^2\Gamma_d$ as $dt\to 0$.

Table 2: Summary of Partial Derivatives (Backward PDEs) in *f*, *V*, *h*, and Γ

Replicating Portfolio Functions	Partial Derivative (Backward Equation)
***f*, density**	$-\frac{\partial f}{\partial t_0}=\frac{1}{2}\sigma^2 S_0^2\frac{\partial^2 f}{\partial S_0^2}+\left(\mu+\frac{1}{2}\sigma^2\right)S_0\frac{\partial f}{\partial S_0}$
***V*, option premium**	$rV-\frac{\partial V}{\partial t_0}=\frac{1}{2}\sigma^2 S_0^2\frac{\partial^2 V}{\partial S_0^2}+\left(\mu+\frac{1}{2}\sigma^2\right)S_0\frac{\partial V}{\partial S_0}$
***h*, option delta**	$\left(r-\mu-\frac{1}{2}\sigma^2\right)h-\frac{\partial h}{\partial t_0}=\frac{1}{2}\sigma^2 S_0^2\frac{\partial^2 h}{\partial S_0^2}+\left(\mu+\frac{3}{2}\sigma^2\right)S_0\frac{\partial h}{\partial S_0}$
Γ, option gamma	$\left(r-2\mu-2\sigma^2\right)\Gamma-\frac{\partial \Gamma}{\partial t_0}=\frac{1}{2}\sigma^2 S_0^2\frac{\partial^2 \Gamma}{\partial S_0^2}+\left(\mu+\frac{5}{2}\sigma^2\right)S_0\frac{\partial \Gamma}{\partial S_0}$

Total Derivatives of Option Related Statistical Functions

The diffusion equation and total derivative of the proportional Brownian motion market asset *S* is

$$\frac{dS_0}{dt_0}=\left(\mu+\tfrac{1}{2}\sigma^2\right)S_0+\sigma S_0\frac{dB_0}{dt_0}$$

This is the stochastic variable and diffusion equation upon which the option and all of its derived functions depend. The diffusion equation of the relevant functions [original to this thesis] of this variable *S* are then obtained from this variable and the relevant backward equation by **(1)** expanding the derivative of the function in a power series around the backward variables (t_0 and S_0), **(2)** separating out o(*dt*) terms, **(3)** substituting in the backward equation, and finally **(4)** cancelling the second order terms. What remains is the diffusion equation, the total derivative of a function of a first order diffusion variable.

TRANSITION DISTRIBUTION OF MARKET ASSET

The transition density *f* of the market asset is

$$f_0=f_{S(T)|S(t_0)}(x\,|\,S_0)=\frac{1}{x\sigma\sqrt{T-t_0}\sqrt{2\pi}}e^{-\frac{[\log x-\log S_0-\mu(T-t_0)]^2}{2\sigma^2(T-t_0)}}$$

which has backward equation

$$-\frac{\partial f_0}{\partial t_0}=\tfrac{1}{2}\sigma^2 S_0^2\frac{\partial^2 f_0}{\partial S_0^2}+\left(\mu+\tfrac{1}{2}\sigma^2\right)S_0\frac{\partial f_0}{\partial S_0}$$

Expanding the transition density $f_0=f_0(t_0,S_0)$ in a power series

$$\begin{aligned}
\frac{df_0}{dt_0}&=\frac{\partial f_0}{\partial t_0}\frac{dt_0}{dt_0}+\frac{\partial f_0}{\partial S_0}\frac{dS_0}{dt_0}+\frac{1}{2}\frac{\partial^2 f_0}{\partial S_0^2}\frac{(dS_0)^2}{dt_0} && \text{expanding in power series}\\
&=\frac{\partial f_0}{\partial t_0}+\frac{\partial f_0}{\partial S_0}\frac{dS_0}{dt_0}+\tfrac{1}{2}\sigma^2 S_0^2\frac{\partial^2 f_0}{\partial S_0^2} && \text{with quadratic variation net of o}(dt)\\
&=-\left(\tfrac{1}{2}\sigma^2 S_0^2\frac{\partial^2 f_0}{\partial S_0^2}+\left(\mu+\tfrac{1}{2}\sigma^2\right)S_0\frac{\partial f_0}{\partial S_0}\right)+\frac{\partial f_0}{\partial S_0}\frac{dS_0}{dt_0}+\tfrac{1}{2}\sigma^2 S_0^2\frac{\partial^2 f_0}{\partial S_0^2} && \text{substituting the PDE}\\
&=-\left(\mu+\tfrac{1}{2}\sigma^2\right)S_0\frac{\partial f_0}{\partial S_0}+\frac{\partial f_0}{\partial S_0}\frac{dS_0}{dt_0} && \text{cancelling } \tfrac{1}{2}\sigma^2 S_0^2\frac{\partial^2 f_0}{\partial S_0^2}
\end{aligned}$$

noting that the diffusion equation of *S* is $dS_0-(\mu+\frac{1}{2}\sigma^2)S_0dt=\sigma S_0 dB_0$ so this can alternatively be written more succinctly in terms of standard Brownian motion dB_0 rather than dS_0

$$\frac{df_0}{dt_0} = -\left(\mu + \tfrac{1}{2}\sigma^2\right) S_0 \frac{\partial f_0}{\partial S_0} + \frac{\partial f_0}{\partial S_0}\frac{dS_0}{dt_0} = \frac{\partial f_0}{\partial S_0}\left(\frac{dS_0}{dt_0} - \left(\mu + \tfrac{1}{2}\sigma^2\right) S_0\right) = \frac{\partial f_0}{\partial S_0}\left(\sigma S_0 \frac{dB_0}{dt_0}\right)$$

i.e.,

$$\frac{df_0}{dt_0} = \sigma S_0 \frac{\partial f_0}{\partial S_0}\frac{dB_0}{dt_0}$$

which is a nice simple expression. This is the diffusion equation of the transition density $f_0 = f_{S(T)|S(t_0)}(x \mid S_0)$.

OPTION PREMIUM

This is my thesis of this book, BOOK 2; The option premium *V* is

$$V_0 = V(k,T;S_0,t_0) = \begin{cases} e^{-r(T-t_0)}\int_k^\infty (x-k) f_{S(T)|S(t_0)}(x \mid S_0)dx \\ e^{-r(T-t_0)}\int_0^k (k-x) f_{S(T)|S(t_0)}(x \mid S_0)dx \end{cases} = \begin{cases} S_0 e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}\overline{F}_Y\left(y_k - \sigma\sqrt{T-t_0}\right) - ke^{-r(T-t_0)}\overline{F}_Y\left(y_k\right) \\ ke^{-r(T-t_0)}F_Y\left(y_k\right) - S_0 e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}F_Y\left(y_k - \sigma\sqrt{T-t_0}\right) \end{cases}$$

which has backward equation

$$-\frac{\partial V_0}{\partial t_0} = \tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 V_0}{\partial S_0^2} + (\mu + \tfrac{1}{2}\sigma^2) S_0 \frac{\partial V_0}{\partial S_0} - rV_0$$

Expanding the option premium $V_0 = V_0(t_0, S_0)$ in a power series

$$\begin{aligned} \frac{dV_0}{dt_0} &= \frac{\partial V_0}{\partial t_0}\frac{dt_0}{dt_0} + \frac{\partial V_0}{\partial S_0}\frac{dS_0}{dt_0} + \tfrac{1}{2}\frac{\partial^2 V_0}{\partial S_0^2}\frac{(dS_0)^2}{dt_0} \\ &= \frac{\partial V_0}{\partial t_0} + \frac{\partial V_0}{\partial S_0}\frac{dS_0}{dt_0} + \tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 V_0}{\partial S_0^2} \\ &= -\left(\tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 V_0}{\partial S_0^2} + \left(\mu + \tfrac{1}{2}\sigma^2\right) S_0 \frac{\partial V_0}{\partial S_0} - rV_0\right) + \frac{\partial V_0}{\partial S_0}\frac{dS_0}{dt_0} + \tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 V_0}{\partial S_0^2} \\ &= rV_0 - \left(\mu + \tfrac{1}{2}\sigma^2\right) S_0 \frac{\partial V_0}{\partial S_0} + \frac{\partial V_0}{\partial S_0}\frac{dS_0}{dt_0} \end{aligned}$$

which is **(1)** the cash rate from decay in the discount factor, **(2)** loss of time value in total market return, and **(3)** market Brownian risk. This total derivative is the continuous time proportional Brownian motion equivalent to the binomial discrete total derivative

$$\frac{\Delta V_{j,n}}{\Delta j} = rV_{j,n} - (pu + qd - 1) S_j h_{j,n} + h_{j,n}\frac{\Delta S_j}{\Delta j} \qquad \text{where} \qquad \frac{\Delta S_j}{\Delta j} = \begin{cases} S_j(u-1) \text{ with probability } p \\ S_j(d-1) \text{ with probability } q \end{cases}$$

In terms of standard Brownian motion dB_0, the total derivative of the option premium for the proportional Brownian motion is

$$\frac{dV_0}{dt_0} = rV_0 + \frac{\partial V_0}{\partial S_0}\left(\frac{dS_0}{dt_0} - \left(\mu + \tfrac{1}{2}\sigma^2\right) S_0\right) = rV_0 + \frac{\partial V_0}{\partial S_0}\left(\sigma S_0 \frac{dB_0}{dt_0}\right)$$

i.e.,

$$\frac{dV_0}{dt_0} = rV_0 + \sigma S_0 \frac{\partial V_0}{\partial S_0}\frac{dB_0}{dt_0}$$

This is the diffusion equation of the option premium $V_0 = V(k,T;S_0,t_0)$ noting that the *dB* term is in proportion to the option delta $h_0 = \partial V_0 / \partial S_0$. The infinitesimal mean of the option premium is rV_0 and the infinitesimal standard deviation is $\sigma S_0 h_0$ which is proportional to the option delta *h* (and the spot price and market volatility).

OPTION DELTA

The option delta *h* is

$$h_0 = h(k,T;S_0,t_0) = \frac{\partial}{\partial S_0}V(k,T;S_0,t_0) = \begin{cases} e^{-r(T-t_0)}\int_k^\infty \frac{x}{S_0} f_{S(T)|S(t_0)}(x \mid S_0)dx \\ e^{-r(T-t_0)}\int_0 \frac{x}{S_0} f_{S(T)|S(t_0)}(x \mid S_0)dx \end{cases} = \begin{cases} e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}\overline{F}_Y\left(y_k - \sigma\sqrt{T-t_0}\right) \\ -e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)}F_Y\left(y_k - \sigma\sqrt{T-t_0}\right) \end{cases}$$

which has backward equation

$$-\frac{\partial h_0}{\partial t_0} = \tfrac{1}{2}\sigma^2 S_0^2\frac{\partial^2 h_0}{\partial S_0^2} + \left(\mu + \tfrac{3}{2}\sigma^2\right)S_0\frac{\partial h_0}{\partial S_0} + \left(\mu + \tfrac{1}{2}\sigma^2 - r\right)h_0 \qquad \text{where} \quad h_0 = \frac{\partial V_0}{\partial S_0}$$

Expanding the option delta $h_0 = h_0(t_0, S_0)$ in a power series

$$\begin{aligned} \frac{dh_0}{dt_0} &= \frac{\partial h_0}{\partial t_0}\frac{dt_0}{dt_0} + \frac{\partial h_0}{\partial S_0}\frac{dS_0}{dt_0} + \tfrac{1}{2}\frac{\partial^2 h_0}{\partial S_0^2}\frac{(dS_0)^2}{dt_0} \\ &= \frac{\partial h_0}{\partial t_0} + \frac{\partial h_0}{\partial S_0}\frac{dS_0}{dt_0} + \tfrac{1}{2}\sigma^2 S_0^2\frac{\partial^2 h_0}{\partial S_0^2} \\ &= -\left(\tfrac{1}{2}\sigma^2 S_0^2\frac{\partial^2 h_0}{\partial S_0^2} + \left(\mu + \tfrac{3}{2}\sigma^2\right)S_0\frac{\partial h_0}{\partial S_0} + \left(\mu + \tfrac{1}{2}\sigma^2 - r\right)h_0\right) + \frac{\partial h_0}{\partial S_0}\frac{dS_0}{dt_0} + \tfrac{1}{2}\sigma^2 S_0^2\frac{\partial^2 h_0}{\partial S_0^2} \\ &= -\left(\mu + \tfrac{1}{2}\sigma^2 - r\right)h_0 - \left(\mu + \tfrac{3}{2}\sigma^2\right)S_0\frac{\partial h_0}{\partial S_0} + \frac{\partial h_0}{\partial S_0}\frac{dS_0}{dt_0} \end{aligned}$$

which can also be written in terms of dB_0 as

$$\frac{dh_0}{dt_0} = -\left(\mu + \tfrac{1}{2}\sigma^2 - r\right)h_0 - \sigma^2 S_0\frac{\partial h_0}{\partial S_0} + \sigma S_0\frac{\partial h_0}{\partial S_0}\frac{dB_0}{dt_0}$$

This is the diffusion equation of the option delta $h_0 = h(k,T;S_0,t_0)$. This shows clearly that a delta update for any option proper depends on gamma and market risk.

OPTION GAMMA

The option gamma Γ is

$$\Gamma_0 = \Gamma(k,T;S_0,t_0) = \frac{\partial^2}{\partial S_0^2}V(k,T;S_0,t_0) = \frac{\partial}{\partial S_0}h(k,T;S_0,t_0)$$

which has backward equation

$$-\frac{\partial \Gamma_0}{\partial t_0} = \tfrac{1}{2}\sigma^2 S_0^2\frac{\partial^2 \Gamma_0}{\partial S_0^2} + \left(\mu + \tfrac{5}{2}\sigma^2\right)S_0\frac{\partial \Gamma_0}{\partial S_0} + \left(2\mu + 2\sigma^2 - r\right)\Gamma_0 \qquad \text{where} \quad \Gamma_0 = \frac{\partial^2 V_0}{\partial S_0^2}$$

Expanding the option gamma $\Gamma_0 = \Gamma_0(t_0, S_0)$ in a power series

$$\begin{aligned} \frac{d\Gamma_0}{dt_0} &= \frac{\partial \Gamma_0}{\partial t_0}\frac{dt_0}{dt_0} + \frac{\partial \Gamma_0}{\partial S_0}\frac{dS_0}{dt_0} + \tfrac{1}{2}\frac{\partial^2 \Gamma_0}{\partial S_0^2}\frac{(dS_0)^2}{dt_0} \\ &= \frac{\partial \Gamma_0}{\partial t_0} + \frac{\partial \Gamma_0}{\partial S_0}\frac{dS_0}{dt_0} + \tfrac{1}{2}\sigma^2 S_0^2\frac{\partial^2 \Gamma_0}{\partial S_0^2} \\ &= -\left(\tfrac{1}{2}\sigma^2 S_0^2\frac{\partial^2 \Gamma_0}{\partial S_0^2} + \left(\mu + \tfrac{5}{2}\sigma^2\right)S_0\frac{\partial \Gamma_0}{\partial S_0} + \left(2\mu + 2\sigma^2 - r\right)\Gamma_0\right) + \frac{\partial \Gamma_0}{\partial S_0}\frac{dS_0}{dt_0} + \tfrac{1}{2}\sigma^2 S_0^2\frac{\partial^2 \Gamma_0}{\partial S_0^2} \\ &= -\left(2\mu + 2\sigma^2 - r\right)\Gamma_0 - \left(\mu + \tfrac{5}{2}\sigma^2\right)S_0\frac{\partial \Gamma_0}{\partial S_0} + \frac{\partial \Gamma_0}{\partial S_0}\frac{dS_0}{dt_0} \end{aligned}$$

which can also be written in terms of dB_0 as

$$\frac{d\Gamma_0}{dt_0} = -\left(2\mu + 2\sigma^2 - r\right)\Gamma_0 - 2\sigma^2 S_0\frac{\partial \Gamma_0}{\partial S_0} + \sigma S_0\frac{\partial \Gamma_0}{\partial S_0}\frac{dB_0}{dt_0}$$

This is the diffusion equation of the option gamma $\Gamma_0 = \Gamma(k,T;S_0,t_0)$.

Table 3: Summary of Total Derivatives (Diffusion Equations) for *f, V, h, Γ*

Replicating Portfolio Functions	Total Derivative in terms of dS_0
***f*, density**	$\frac{df_0}{dt_0} = -\left(\mu + \frac{1}{2}\sigma^2\right)S_0\frac{\partial f_0}{\partial S_0} + \frac{\partial f_0}{\partial S_0}\frac{dS_0}{dt_0}$
***V*, option premium**	$\frac{dV_0}{dt_0} = rV_0 - \left(\mu + \frac{1}{2}\sigma^2\right)S_0\frac{\partial V_0}{\partial S_0} + \frac{\partial V_0}{\partial S_0}\frac{dS_0}{dt_0}$
***h*, option delta**	$\frac{dh_0}{dt_0} = -\left(\mu + \frac{1}{2}\sigma^2 - r\right)h_0 - \left(\mu + \frac{3}{2}\sigma^2\right)S_0\frac{\partial h_0}{\partial S_0} + \frac{\partial h_0}{\partial S_0}\frac{dS_0}{dt_0}$
Γ, option gamma	$\frac{d\Gamma_0}{dt_0} = -\left(2\mu + 2\sigma^2 - r\right)\Gamma_0 - \left(\mu + \frac{5}{2}\sigma^2\right)S_0\frac{\partial \Gamma_0}{\partial S_0} + \frac{\partial \Gamma_0}{\partial S_0}\frac{dS_0}{dt_0}$

Table 4: Summary of Total Derivatives (in terms of *dB*) *f, V, h, Γ*

Replicating Portfolio Functions	Total Derivative in terms of dB_0
***f*, density**	$\frac{df_0}{dt_0} = \sigma S_0\frac{\partial f_0}{\partial S_0}\frac{dB_0}{dt_0}$
***V*, option premium**	$\frac{dV_0}{dt_0} = rV_0 + \sigma S_0\frac{\partial V_0}{\partial S_0}\frac{dB_0}{dt_0}$
***h*, option delta**	$\frac{dh_0}{dt_0} = -\left(\mu + \frac{1}{2}\sigma^2 - r\right)h_0 - \sigma^2 S_0\frac{\partial h_0}{\partial S_0} + \sigma S_0\frac{\partial h_0}{\partial S_0}\frac{dB_0}{dt_0}$
Γ, option gamma	$\frac{d\Gamma_0}{dt_0} = -\left(2\mu + 2\sigma^2 - r\right)\Gamma_0 - 2\sigma^2 S_0\frac{\partial \Gamma_0}{\partial S_0} + \sigma S_0\frac{\partial \Gamma_0}{\partial S_0}\frac{dB_0}{dt_0}$

Components of Change in the Option Replicating Portfolio for the Brownian Motion Market Asset S

Asset/Liability		Quantity	Value
market asset,	S_t	$+h_t$	$+h_tS_t$
option,	V_t	-1	$-V_t$
total,	Π_t		$h_tS_t - V_t$

The cost of establishing the portfolio is **(1)** the cost of the offsetting asset, the initial delta times the underlying asset price, less **(2)** the premium revenue from writing the option

$$\Pi_{0,T} = h_{0,T}S_0 - V_{0,T}$$

If the portfolio is then rebalanced infinitely often so that it is updated continuously, and consequently the re-balancing maintains a continuous stochastic signal that replicates the instantaneous option delta and continually remains matched, the total units of the market asset transacted to rebalance is $\int_0^t dh_{y,T} = h_{t,T} - h_{0,T}$ units in addition to the $h_{0,T}$ market asset purchased at the portfolios establishment. The portfolio value at time *t* is

$$\begin{aligned}\Pi_{t,T} &= (h_{0,T} + \int_0^t dh_{0,T})S_t - V_{t,T}\\ &= (h_{0,T} + h_{t,T} - h_{t,T})S_t - V_{t,T}\\ &= h_{t,T}S_t - V_{t,T}\end{aligned}$$

The total continuous cost of rebalancing the replicating delta by time *t* is therefore $\int_0^t S_y dh_{y,T}$ which is the continuous sum of each of the infinitely transacted buys and sells in the market asset at quantity amounts (rather than dollar amounts) specified by the instantaneous delta changes. The present value of these delta updates is then the integral of discounted value of the continuous stream of the instantaneous transaction costs

$$\int_0^t e^{-ry} S_y dh_{y,T}$$

When integrated to the end of the deferral period at time *T*, this term is the present value of all of the portfolio cashflows between the establishment of the portfolio at time zero and the closing out of the portfolio at option expiry.

When the rebalances are made continuously often, and hence maintain a continuous signal, the portfolio remains delta-neutral to option expiry. Noting that the call and put option premiums can be written in terms of delta as

$$\begin{aligned} V_{t,T} &= S_T e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t)} \overline{F}_U\left(u_k - \sigma\sqrt{T-t}\right) - ke^{-r(T-t)} \overline{F}_U\left(u_k\right) \\ &= S_t h_{t,T} - ke^{-r(T-t)} \overline{F}_U\left(u_k\right) \\ &= S_t h_{t,T} - ke^{-r(T-t)} \Pr\left(S(T) > k \mid S(t) = S_t\right) \end{aligned}$$

$$\begin{aligned} V_{t,T} &= ke^{-r(T-t)} F_U\left(u_k\right) - S_t e^{(\mu-r+\frac{1}{2}\sigma^2)(T-t)} F_U\left(u_k - \sigma\sqrt{T-t}\right) \\ &= ke^{-r(T-t)} F_U\left(u_k\right) + S_t h_{t,T} \\ &= ke^{-r(T-t)} \Pr\left(S(T) < k \mid S(t) = S_t\right) + S_t h_{t,T} \end{aligned}$$

The delta-neutral replicating portfolio $\Pi_{t,T} = h_{t,T} S_t - V_{t,T}$ is then the discounted value of the exercise price times the option moneyness probability

$$\begin{aligned} \Pi_{t,T} &= h_{t,T} S_t - \left(S_t h_{t,T} - ke^{-r(T-t)} \overline{F}_U\left(u_k\right)\right) \\ &= ke^{-r(T-t)} \overline{F}_U\left(u_k\right) \\ &= ke^{-r(T-t)} \Pr\left(S(T) > k \mid S(t) = S_t\right) \end{aligned}$$

for a call option portfolio

$$\begin{aligned} \Pi_{t,T} &= h_{t,T} S_t - \left(ke^{-r(T-t)} F_U\left(u_k\right) + S_t h_{t,T}\right) \\ &= -ke^{-r(T-t)} F_U\left(u_k\right) \\ &= -ke^{-r(T-t)} \Pr\left(S(T) < k \mid S(t) = S_t\right) \end{aligned}$$

for a put option portfolio

Since these are just discounted CDFs and complementary CDFS, and hence have the same dynamics as the probability distribution of the market asset [with a source] and hence also that of the option premium, the delta-neutral replicating portfolio then has a backward equation and diffusion equation exactly that of the option [but obviously with a different dependent variable, Π rather than *V*]. Provided that updates are made continuously, these are

$$r\Pi_{t,T} - \frac{\partial \Pi_{t,T}}{\partial t} = \tfrac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 \Pi_{t,T}}{\partial S_t^2} + (\mu + \tfrac{1}{2}\sigma^2) S_t \frac{\partial \Pi_{t,T}}{\partial S_t}$$

and

$$\frac{d\Pi_{t,T}}{dt_0} = r\Pi_{t,T} + \sigma S_t \frac{\partial \Pi_{t,T}}{\partial S_t} \frac{dB_t}{dt}$$

which shows that the delta-neutral portfolio earns the cash rate [from time decay in the discount factor] plus market Brownian risk [from the dependence of the CDFs and complementary CDFs on the diffusion variable *S*] and hence has similar risk as the option itself. This risk is paid for with the rebalances for the delta-updates, where the deltas of an option market for which prices are not debased carry the market return for the market. Note that by differentiating the delta-neutral portfolio partially with respect to the market spot price

$$\frac{\partial}{\partial S_t}\Pi_{t,T} = \frac{\partial}{\partial S_t}\left(h_{t,T} S_t - V_{t,T}\right) = h_{t,T}\frac{\partial}{\partial S_t} S_t + S_t \frac{\partial}{\partial S_t} h_{t,T} - \frac{\partial}{\partial S_t} V_{t,T} = h_{t,T} + S_t \frac{\partial h_{t,T}}{\partial S_t} - \frac{\partial V_{t,T}}{\partial S_t} = S_t \Gamma_{t,T}$$

the diffusion equation of the delta-neutral replicating portfolio can be expressed in terms of gamma as

$$\frac{d\Pi_{t,T}}{dt_0} = r\Pi_{t,T} + \sigma {S_t}^2 \Gamma_{t,T} \frac{dB_t}{dt}$$

showing again that any break in the continuous stream of *dB* dependent delta rebalances, paid for out of inflows and outflows from equity, introduces gamma variation into the net portfolio cashflows.

The portfolio cashflows consist of three parts. The first is the transaction establishing the portfolio, consisting of the exchange of option premium and the cost of taking the offsetting position in the market asset. The second is the stream of delta rebalances. The remaining item of cashflow is the value of closing out the portfolio when the option expires. This is the boundary condition of the delta-neutralized replicating portfolio. If the option is outofthemoney, then both the option and the delta are zero and the portfolio value is zero.

In the case of a call option, if the option is inthemoney, the delta is one and the portfolio is long one unit of the market asset for the long spot trade. The option must be delivered which then has a net value of the excess of the asset over the exercise price. The net portfolio value and closing out proceeds is then the exercise price

$$\Pi_{T,T} = \begin{cases} 0 \times S_T - V_{T,T} \\ 1 \times S_T - V_{T,T} \end{cases} = \begin{cases} 0 - 0 \\ S_T - (S_T - k) \end{cases} = \begin{cases} 0 & \text{if outofthemoney} \\ k & \text{if inthemoney} \end{cases}$$

In the case of a put option, if the option is inthemoney, the delta is negative one and the portfolio is short one unit of the market asset for the short spot trade. The option must be delivered which then has a net value of the shortfall of the asset from the exercise price. The portfolio value and closing out proceeds is then a negative value of the exercise price

$$\Pi_{T,T} = \begin{cases} 0 \times S_T - V_{T,T} \\ -1 \times S_T - V_{T,T} \end{cases} = \begin{cases} 0 - 0 \\ -S_T - (k - S_T) \end{cases} = \begin{cases} 0 & \text{if outofthemoney} \\ -k & \text{if inthemoney} \end{cases}$$

It is now proved, by deriving the return components of the changes in the portfolio, that what remains [from net portfolio cashflows] in the case of the accurate premium replicated and offset by the market asset, is then return on market assets which are traded to take up the pecuniary interest in the economic activity of the underlying market asset for the spot transaction that is deferred to the option maturity date. This is the economic value of the derivative, and is in direct proportion to the option delta and the path of the underlying market asset. The constant of proportionality is the rate of market return of the market asset net of cash. This is positive for a call option, which **takes up the pecuniary interest in the economic activity** of the market asset for the deferred transaction with a long position, and negative for a put option, which **shorts the pecuniary interest in the economic activity** of the market asset for the deferred transaction with a short position. This return is what you get in return for flatlining the option position, taking a pecuniary interest in the underlying spot market, and doing a whole heap of asset trading. In particular, it is the **revenue earned** for the **financial service** of **making a market in the option** and **taking up the pecuniary interest** in the **economic activity of the market asset** for the **deferred transaction** of the derivative that you have made a market in; long in *S* for a call and short in *S* for a put.

Table 5

Replicating Portfolio Functions	Formulae
Call option premium, *V*	$V(k,T;S_0,t_0) = S_0 e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)} \overline{F}_U\left(u_k - \sigma\sqrt{T-t_0}\right) - k e^{-r(T-t_0)} \overline{F}_U\left(u_k\right)$
Call option delta, *h*	$h(k,T;S_0,t_0) = e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)} \overline{F}_U\left(u_k - \sigma\sqrt{T-t_0}\right)$
delta neutralized call option portfolio, Π	$\Pi(k,T;S_0,t_0) = k e^{-r(T-t_0)} \overline{F}_U\left(u_k\right)$
Put option premium, *V*	$V(k,T;S_0,t_0) = k e^{-r(T-t_0)} F_U\left(u_k\right) - S_0 e^{(\mu-r+\frac{1}{2}\sigma^2)(T-t_0)} F_U\left(u_k - \sigma\sqrt{T-t_0}\right)$
Put option delta, *h*	$h(k,T;S_0,t_0) = -e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t_0)} F_U\left(u_k - \sigma\sqrt{T-t_0}\right)$
Delta neutralized put option portfolio, Π	$\Pi(k,T;S_0,t_0) = -k e^{-r(T-t_0)} F_U\left(u_k\right)$

The Return Components of the Replicating Portfolio of the Statistically Unbiased Option for the Market Asset S

The total derivative of the option replicating portfolio $\Pi_t = \Delta_t S_t - V_t$ is

$$\frac{d\Pi_t}{dt} = h_t \frac{dS_t}{dt} - \frac{dV_t}{dt} + (S_t + dS_t)\frac{dh_t}{dt} \qquad \text{where} \quad h_t = \frac{\partial V_t}{\partial S_t}$$

which is the change in the value of the market asset held in the portfolio in proportion to delta, the change in the value of the short option, and the change in the replicating delta which must be rebalanced with an update; a buy or sell in the market asset. This expression is now proved by several alternative methods to be

$$\frac{d\Pi_t}{dt} = (\mu + \tfrac{1}{2}\sigma^2)S_t h_t - rV_t + (S_t + dS_t)\frac{dh_t}{dt}$$

which says that the portfolio earns the market rate of return on the delta position in the market asset and incurs the cash rate on the short option, net of the instantaneous delta rebalance outstanding, i.e.,

$$h_{t,T}\frac{dS_t}{dt} - \frac{dV_{t,T}}{dt} = (\mu + \tfrac{1}{2}\sigma^2)S_t h_{t,T} - rV_{t,T}$$

This expression is dependent on the path of delta since the portfolio returns are not fixed flat, but instead vary in two parts; one in proportion to the quantity of the market asset held *hS*, which earns the market rate of return, and the other in proportion to the value of the option *V*, which keeps up with cash from decay in the discount factor. This results in **convex portfolio flux** in market return since the value of the market asset held, and the rate at which it earns, both exceed that of the option ($hS > V$ and $\mu + \tfrac{1}{2}\sigma^2 > r$). The portfolio returns are only fixed flat at cash if the portfolio has been debased (and convexity zeroed) with a debased premium that is accurate only for oxymoron risky market assets that have no market return. The mispricing in the premium then kills the return on the market asset, whatever it may earn, leaving the portfolio return and all portfolio cashflows debased and void of market return.

Note that expression for portfolio returns for the portfolio replicating the accurate premium is equivalent to the binomial expression

$$\frac{\Delta\Pi_{j,n}}{\Delta j} = (pu + qd - 1)S_j h_{j,n} - rf_{j,n} + S_{j+1}\frac{\Delta h_{j,n}}{\Delta j}$$

The fact that a delta-neutralized portfolio does not earn the cash rate [unless it has been debased], net of a delta update, but instead earns the market rate of return on the market asset and incurs the cash rate on the option, and hence the expression for the return components of the option portfolio replicating the accurate premium $h_{t,T}dS_t - dV_{t,T} = (\mu + \tfrac{1}{2}\sigma^2)S_t h_{t,T}dt - rV_{t,T}dt$, is deduced and hence proved by three methods. These are

- **(1)** statistical functions (mean, variance, and covariance) of the diffusion equations of the market asset and the option premium
- **(2)** taking expectations and substituting in the backward deconvolution integral for the premium
- **(3)** differentiating the portfolio and substituting in the total derivative of the option premium

which, aside from the combinatorial proof [which is not possible in continuous time], correspond one-for-one with the methods for $h_{j,n}\Delta S_{j,n} - \Delta f_{j,n} = (pu + qd - 1)S_j h_{j,n} - r\, f_{j,n}$ in the binomial context. Each of these three proofs stress different technicalities of how the replicating portfolio earns return [although are essentially the same thing].

Delta neutral portfolios earn market return on market asset and cash rate on option

Proof 1

The replicating portfolio is

$$\Pi_t = h_t S_t - V_t$$

where $h_t S_t$ replicates the delta of the option V_t. Differentiating the portfolio whilst holding the replicating delta constant

$$d\Pi_t = h_t dS_t - dV_t$$

The offsetting delta position in the replicating asset offsets and kills the instantaneous option delta over the infinitesimal transition *t* to *t* + *dt* and nets out the Gaussian basis function for that infinitesimal instant. Therefore it suffices to show return components by deducing the expectation of the change. Rigor requires it also to be

shown that the variance is zero $Var\left(d\Pi_t \mid \Pi_t\right)=0$ which this thesis does, as for the binomial walk in PART I. This is interesting for it shows what the variance of each of the two items are and how these two items covary.

The diffusion equation of the market asset *S* is

$$dS_t = (\mu + \tfrac{1}{2}\sigma^2)S_t dt + \sigma S_t dB_t$$

for which the infinitesimal mean and variance are

$$E\left(dS_t\right) = (\mu + \tfrac{1}{2}\sigma^2)S_t dt \qquad Var\left(dS_t\right) = \sigma^2 S_t^{\,2} dt$$

The diffusion equation of the option premium *V* [original to this thesis] is

$$dV_t = rV_t dt + \sigma S_t h_t dB_t$$

for which the infinitesimal mean and variance are

$$E\left(dV_t\right) = rV_t dt \qquad Var\left(dV_t\right) = \sigma^2 S_t^{\,2} h_t^{\,2} dt$$

With the understanding that all statistical functions are conditional on the time *t* spot price S_t, the expectation of the differential change in the replicating portfolio is

$$\begin{aligned} E\left(d\Pi_t\right) &= E\left(h_t dS_t - dV_t\right) \\ &= h_t E\left(dS_t\right) - E\left(dV_t\right) \\ &= (\mu + \tfrac{1}{2}\sigma^2)h_t S_t dt - rV_t dt \end{aligned}$$

and the proof is complete, market return on the market asset and cash rate on the option. For rigor, the variance of the portfolio is now proved to be zero. The covariance between the replicating delta position in the asset and the option premium *V* is

$$\begin{aligned} Cov\left(dS_t, dV_t\right) &= Cov\left((\mu + \tfrac{1}{2}\sigma^2)S_t dt + \sigma S_t dB_t,\ rV_t dt + \sigma S_t h_t dB_t\right) \\ &= Cov\left(\sigma S_t dB_t, \sigma S_t h_t dB_t\right) \\ &= \sigma^2 S_t^{\,2} h_t Cov\left(dB_t, dB_t\right) \\ &= \sigma^2 S_t^{\,2} h_t Var\left(dB_t\right) \\ &= \sigma^2 S_t^{\,2} h_t dt \end{aligned}$$

and hence, with $Var\left(dS_t\right) = \sigma^2 S_t^{\,2} dt$ and $Var\left(dV_t\right) = \sigma^2 S_t^{\,2} h_t^{\,2} dt$, the variance of the differential change in the portfolio is

$$\begin{aligned} Var\left(d\Pi_t\right) &= Var\left(h_t dS_t - dV_t\right) \\ &= h_t^{\,2} Var\left(dS_t\right) + Var\left(dV_t\right) - 2h_t Cov\left(dS_t, dV_t\right) \\ &= h_t^{\,2} Var\left(dS_t\right) + Var\left(dV_t\right) - 2h_t Cov\left(dS_t, dV_t\right) \\ &= h_t^{\,2}\left(\sigma^2 S_t^{\,2} dt\right) + \left(\sigma^2 S_t^{\,2} h_t^{\,2} dt\right) - 2h_t\left(\sigma^2 S_t^{\,2} h_t dt\right) \\ &= 0 \end{aligned}$$

Hence, the return on **the portfolio free to earn return** earns the market rate on the market asset and incurs the cash rate on the short option, rigorously, QED.

Proof 2

Holding the option delta *h* constant over the infinitesimal transition interval from *t* to *t* + *dt* (to be paid at *t* + *dt*)

$$
\begin{aligned}
d\Pi_t &= E\left(d\Pi_t \mid \Pi_t\right) \\
&= E\left(\Pi(k,T;S_t + dS_t, t+dt) - \Pi(k,T;S_t,t)\right) \\
&= E\left(\left(h_t(S_t + dS_t) - V(k,T;S_t + dS_t, t+dt)\right) - \left(h_t S_t - V(k,T;S_t,t)\right)\right) \\
&= E\left(h_t dS_t - \left(V(k,T;S_t + dS_t, t+dt) - V(k,T;S_t,t)\right)\right) \\
&= \int_0^\infty f_X(x)\left\{\left(h_t S_t \times x - h_t S_t\right) - \left(V(k,T;S_t \times x, t+dt) - V(k,T;S_t,t)\right)\right\} dx \\
&= \left(\int_0^\infty x f_X(x) dx - 1\right) h_t S_t - \left(\int_0^\infty f_X(x) V(k,T;S_t \times x, t+dt) dx - V(k,T;S_t,t)\right) \\
&= \left(e^{(\mu + \frac{1}{2}\sigma^2)dt} - 1\right) h_t S_t - \left(e^{rdt} \int_0^\infty f_X(x) e^{-rdt} V(k,T;S_t \times x, t+dt) dx - V(k,T;S_t,t)\right)
\end{aligned}
$$

where *f* is the log-normal basis density of the market asset

$$f_X(x) = \frac{1}{x\sigma\sqrt{2\pi\, dt}} e^{-\frac{1}{2}\frac{(\log x - \mu dt)^2}{\sigma^2 dt}}$$

Substituting in the backward recurrence of the option $\int_0^\infty f_X(x) e^{-rdt} V(k,T;S_t \times x, t+dt) dx = V(k,T;S_t,t)$

$$
\begin{aligned}
d\Pi_t &= \left(e^{(\mu + \frac{1}{2}\sigma^2)dt} - 1\right) h_t S_t - \left(e^{rdt} V(k,T;S_t,t) - V(k,T;S_t,t)\right) \\
&= \left(e^{(\mu + \frac{1}{2}\sigma^2)dt} - 1\right) h_t S_t - \left(e^{rdt} - 1\right) V(k,T;S_t,t) \\
&= \left(\mu + \tfrac{1}{2}\sigma^2\right) h_t S_t dt - rV(k,T;S_t,t) dt + o(dt)
\end{aligned}
$$

which is the market rate of return on the market asset and the cash rate on the short option. Dividing through by *dt*

$$\frac{d\Pi_t}{dt} = \left(\mu + \tfrac{1}{2}\sigma^2\right) h_t S_t - rV_t + \frac{o(dt)}{dt}$$

Taking limits by letting $dt \to 0$

$$\frac{d\Pi_t}{dt} = \left(\mu + \tfrac{1}{2}\sigma^2\right) h_t S_t - rV_t$$

It is only flat at the cash rate if the portfolio has been debased; either **(1)** the basis variable *X* has been debased with a slack variable Δ [which means a debased premium replicated by a demeaned asset *Y*] or **(2)** the premium has been debased which kills the returns earned with the original basis components [if the replicating asset has been remeaned whilst leaving the premium debased]. This, for either scenario, occurs surreptitiously by holding the portfolio return flat at $d\Pi_t = r\Pi_t dt$. This is then the condition that characterizes debased option portfolios, $d\Pi_t^{debased} = r\Pi^{debased} dt$, and option derivatives as redundant securities of no economic value.

Proof 3

This proof is whereby the total derivative of the option premium is substituted into the portfolio derivative [which is essentially the same steps to derive the diffusion equation of the option premium]. From the portfolio value at time *t* ,

$$\Pi_t = S_t h_t - V_t \qquad \text{where} \quad h_t = \frac{\partial V_t}{\partial S_t}$$

the time derivative is, by the product rule,

$$\frac{d\Pi_t}{dt} = (S_t + dS_t)\frac{dh_t}{dt} + h_t \frac{dS_t}{dt} - \frac{dV_t}{dt}$$

By the total derivative of a variable with two independent variables, time and a stochastic variable of a first order normal diffusion, expanding the option premium $V = V(S,t)$ in a power series around the backward variables

$$\frac{d\Pi_t}{dt} = (S_t + dS_t)\frac{dh_t}{dt} + h_t\frac{dS_t}{dt} - \left(\frac{\partial V_t}{\partial t}\frac{dt}{dt} + \frac{\partial V_t}{\partial S_t}\frac{dS_t}{dt} + \tfrac{1}{2}\frac{\partial^2 V_t}{\partial S_t^{\,2}}\frac{(dS_t)^2}{dt}\right)$$

and cancelling the replicating delta with the instantaneous option delta $h_t = \frac{\partial V_t}{\partial S_t}$

$$\frac{d\Pi_t}{dt} = (S_t + dS_t)\frac{dh_t}{dt} - \frac{\partial V_t}{\partial t} - \tfrac{1}{2}\frac{\partial^2 V_t}{\partial S_t^{\,2}}\frac{(dS_t)^2}{dt}$$

The diffusion equation of the market asset is $dS_t = (\mu + \frac{1}{2}\sigma^2)\,S_t\,dt + \sigma S_t\,dB_t$ for $dB_t \sim N(0, dt)$ and the square of the stochastic term is $(dB_t)^2 \sim \chi_1^{\,2}\,dt \to dt + o(dt)$ giving $(dS_t)^2 = \sigma^2 S_t^{\,2} dt + o(dt)$. The time derivative of the portfolio becomes, net of o(dt) variation,

$$\frac{d\Pi_t}{dt} = (S_t + dS_t)\frac{dh_t}{dt} - \frac{\partial V_t}{\partial t} - \tfrac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V_t}{\partial S_t^{\,2}}$$

and substituting the partial derivative in the option premium from the PDE of the statistically unbiased premium $-\partial_t V_t = \frac{1}{2}\sigma^2 S_t \partial^2_{SS} V_t + (\mu + \frac{1}{2}\sigma^2) S_t \partial_S V_t - rV_t$ into the portfolio time derivative

$$\frac{d\Pi_t}{dt} = (S_t + dS_t)\frac{dh_t}{dt} + \left(\tfrac{1}{2}\sigma^2 S_t\frac{\partial^2 V_t}{\partial S_t^2} + (\mu + \tfrac{1}{2}\sigma^2)S_t\frac{\partial V_t}{\partial S_t} - rV_t\right) - \tfrac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V_t}{\partial S_t^{\,2}}$$

Cancelling the quadratic terms in the option gamma $\frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V_t}{\partial S_t^{\,2}}$ leaves the expression

$$\frac{d\Pi_t}{dt} = (S_t + dS_t)\frac{dh_t}{dt} + (\mu + \tfrac{1}{2}\sigma^2)S_t\frac{\partial V_t}{\partial S_t} - rV_t$$

Net of the instantaneous change in the delta and hence the update for delta, this is

$$\frac{d\Pi_t}{dt} = (\mu + \tfrac{1}{2}\sigma^2)S_t\frac{\partial V_t}{\partial S_t} - rV_t$$

which is the market return on the market asset and the cash rate on the short option. This is in contrast to the portfolio time derivative of a debased portfolio [of return-neutral valuation] whereby the value changes of the portfolio are held fixed flat at the cash rate [and killed of convex flux in market return].

In short,

(A) The total derivative of the option is $\frac{dV_t}{dt} = rV_t - (\mu + \frac{1}{2}\sigma^2)S_t\frac{\partial V_t}{\partial S_t} + \frac{\partial V_t}{\partial S_t}\frac{dS_t}{dt}$

Differentiating the portfolio $\Pi_{t,T} = h_{t,T}S_t - V_{t,T}$

$$\frac{d\Pi_{t,T}}{dt} = h_{t,T}\frac{dS_t}{dt} - \frac{dV_{t,T}}{dt} + (S_t + dS_t)\frac{dh_{t,T}}{dt}$$

which is composed of a change in the value of the replicating asset, less a change in the value of the option, plus an delta rebalance. Substituting in the total derivative of the option

$$\begin{aligned}\frac{d\Pi_{t,T}}{dt} &= h_{t,T}\frac{dS_t}{dt} - \left(rV_{t,T} - (\mu + \tfrac{1}{2}\sigma^2)S_t h_{t,T} + h_{t,T}\frac{dS_t}{dt}\right) + (S_t + dS_t)\frac{dh_{t,T}}{dt} \\ &= (\mu + \tfrac{1}{2}\sigma^2)S_t h_{t,T} - rV_{t,T} + (S_t + dS_t)\frac{dh_{t,T}}{dt}\end{aligned}$$

by canceling the replicating delta with the option delta. Net of the update this is

$$\frac{d\Pi_{t,T}}{dt} = (\mu + \tfrac{1}{2}\sigma^2)S_t h_{t,T} - rV_{t,T}$$

which shows that the portfolio earns the market rate of return on the market asset and incurs the cash rate on the short option $d\Pi_{t,T} = (\mu + \frac{1}{2}\sigma^2)S_t h_{t,T} dt - rV_{t,T} dt$.

(B) The total derivative and diffusion equation of the option is alternatively written in terms of standard Brownian motion as

$$\frac{dV_t}{dt} = rV_t + \sigma S_t \frac{\partial V_t}{\partial S_t}\frac{dB_t}{dt}$$

The diffusion equation of the asset is

$$\frac{dS_t}{dt} = (\mu + \tfrac{1}{2}\sigma^2)S_t + \sigma S_t \frac{dB_t}{dt}$$

so that delta units of this is

$$\frac{\partial V_t}{\partial S_t}\frac{dS_t}{dt} = (\mu + \tfrac{1}{2}\sigma^2)S_t \frac{\partial V_t}{\partial S_t} + \sigma S_t \frac{\partial V_t}{\partial S_t}\frac{dB_t}{dt}$$

Substituting these into the total portfolio derivative

$$\begin{aligned}\frac{d\Pi_{t,T}}{dt} &= h_{t,T}\frac{dS_t}{dt} - \frac{dV_{t,T}}{dt} + (S_t + dS_t)\frac{dh_{t,T}}{dt} \\ &= \left((\mu + \tfrac{1}{2}\sigma^2)S_t \frac{\partial V_t}{\partial S_t} + \sigma S_t \frac{\partial V_t}{\partial S_t}\frac{dB_t}{dt}\right) - \left(rV_t + \sigma S_t \frac{\partial V_t}{\partial S_t}\frac{dB_t}{dt}\right) + (S_t + dS_t)\frac{dh_{t,T}}{dt} \\ &= (\mu + \tfrac{1}{2}\sigma^2)S_t \frac{\partial V_t}{\partial S_t} - rV_t + (S_t + dS_t)\frac{dh_{t,T}}{dt}\end{aligned}$$

upon cancelling the two dB_t terms. The result is as before showing that, $d\Pi_{t,T} = (\mu + \frac{1}{2}\sigma^2)h_{t,T}S_t dt - rV_{t,T} dt$ net of the instantaneous delta update due, showing [clearly] that the portfolio earns the market rate of return on the market asset $E(h_t dS_t \mid S_t) = (\mu + \frac{1}{2}\sigma^2)h_t S_t dt$ and incurs the cash rate on the short option $E(dV_t \mid S_t) = rV_t dt$., i.e., shows clearly that the infinitesimal mean and market return on the market asset is $\mu + \frac{1}{2}\sigma^2$ and that the infinitesimal mean and return on the option is the cash rate *r*. The cash rate on the option premium is simply a result of the discount factor $e^{-r(T-t_0)}$ placed on the premium to put it into present values. The portfolio only earns a flat rate at cash, null of market return, if either the asset earns nothing or if the option premium has been debased with a slack variable which kills the market return earned from taking a position in the market asset [to flatline the option by killing its delta].

The proof is complete. Quod Erat Demonstrandum, as was to be demonstrated.

An o(dt) Expression for the Instantaneous Delta Update for the Replicating Portfolio for the Statistically Unbiased Option and Market Asset S

The total derivatives of the market asset and the instantaneous option delta are

$$dS_t = \left(\mu + \tfrac{1}{2}\sigma^2\right)S_t dt + \sigma S_t dB_t \qquad \text{and} \qquad dh_t = -\left(\mu + \tfrac{1}{2}\sigma^2 - r\right)h_t dt - \sigma^2 S_t \frac{\partial h_t}{\partial S_t}dt + \sigma S_t \frac{\partial h_t}{\partial S_t}dB_t$$

Hence, the product of the two stochastic differentials is

$$
\begin{aligned}
dS_t dh_t &= \left(\left(\mu + \tfrac{1}{2}\sigma^2\right)S_t dt + \sigma S_t dB_t\right)\left(-\left(\mu + \tfrac{1}{2}\sigma^2 - r\right)h_t dt - \sigma^2 S_t \frac{\partial h_t}{\partial S_t} dt + \sigma S_t \frac{\partial h_t}{\partial S_t} dB_t\right) \\
&= \sigma S_t \left(\sigma S_t \frac{\partial h_t}{\partial S_t}\right)(dB_t)^2 - \left(\mu + \tfrac{1}{2}\sigma^2\right)S_t\left(\left(\mu + \tfrac{1}{2}\sigma^2 - r\right)h_t + \sigma^2 S_t \frac{\partial h_t}{\partial S_t}\right)(dt)^2 \\
&\qquad + \left(\mu + \tfrac{1}{2}\sigma^2\right)\sigma {S_t}^2 \frac{\partial h_t}{\partial S_t}(dB_t dt) - \sigma S_t\left(\left(\mu + \tfrac{1}{2}\sigma^2 - r\right)h_t dt + \sigma^2 S_t \frac{\partial h_t}{\partial S_t} dt\right)(dB_t dt) \\
&= \sigma^2 S_t^2 \frac{\partial h_t}{\partial S_t} dt + o(dt) \\
&= \sigma^2 S_t^2 \Gamma_t dt + o(dt) \qquad \text{where } \Gamma_t = \frac{\partial h_t}{\partial S_t}
\end{aligned}
$$

since $(dt)^2 = o(dt)$, $(dB_t dt) = o(dt)$, and $(dB_t)^2 = dt + o(dt)$, and where the gamma is the partial rate of change in the delta *h* with respect to the spot price of the market asset *S*. Hence, the instantaneous rebalance of the replicating delta can be written as

$$
\begin{aligned}
(S_t + dS_t)\frac{dh_t}{dt} &= S_t \frac{dh_t}{dt} + \frac{dh_t}{dt} dS_t + o(dt) \\
&= S_t \frac{dh_t}{dt} + \sigma^2 S_t^2 \frac{\partial h_t}{\partial S_t}\frac{dh_t}{dt} + o(dt) \\
&= \left(S_t + \sigma^2 S_t^2 \Gamma_t\right)\frac{dh_t}{dt} + o(dt)
\end{aligned}
$$

which makes it clear that one break in rebalancing the instantaneous replicating delta (one missed *dt*) means that gamma enters into the portfolio cashflows from which it cannot be removed or neutralized. This means that any replicating portfolio for any option proper in real markets always has gamma risk. It is only free from gamma if the portfolio achieves **full immunization**, say for a pseudo option or a future, since then and only then the delta total derivative and the rebalances are not statistically variable and are independent of gamma.

The total differential change in the replicating portfolio can be expressed net of o(*dt*) variation as

$$\frac{d\Pi_{t,T}}{dt} = (\mu + \tfrac{1}{2}\sigma^2)h_{t,T}S_t - rV_{t,T} + (S_t + \sigma^2 S_t^2 \Gamma_{t,T})\frac{dh_{t,T}}{dt}$$

Note that it is not possible to invest in the square of the market asset (which is always the coefficient of gamma), nor any other power of the market asset (required for any other higher order hedge ratio), so that the replicating portfolio is still notionally rebalanced [infinitely often and hence continuously] with $(S_t + dS_t)dh_{t,T}$.

NPV of the Cashflows of the Replicating Portfolio for the Statistically Unbiased Option and Market Asset S

The statistically accurate option premium,

$V_{t,T} = e^{-r(T-t)}E\left(\max(S(T)-k,0)\,|\,S(t)=S_t\right)$ for a call option or

$V_{t,T} = e^{-r(T-t)}E\left(\max(k-S(T),0)\,|\,S(t)=S_t\right)$ for a put option

is replicated and offset by trading the underlying market asset *S* in proportion to the instantaneous option delta to offset the option delta and flatline the payoff profile of the option. The return components plus the delta update together compose the total time derivative of the replicating portfolio

$$\frac{d}{dt}\left(S_t h_t - V_t\right) = (\mu + \tfrac{1}{2}\sigma^2)S_t h_t - rV_t + (S_t + \sigma^2 S_t^2 \Gamma_{t,T})\frac{dh_t}{dt} \qquad \text{where } h_t = \frac{\partial V_t}{\partial S_t}$$

net of o(*dt*) variation. This is a first order linear differential equation describing the time dynamics of the delta quantity of the market asset and the time dynamics of the option, with a single input of the continuous delta rebalances. Taking $rS_t h_t$ from both sides and rearranging

$$\frac{d}{dt}\left(S_t h_t - V_t\right) - r\left(S_t h_t - V_t\right) - (S_t + \sigma^2 S_t^2 \Gamma_{t,T})\frac{dh_t}{dt} = (\mu + \tfrac{1}{2}\sigma^2 - r)S_t h_t$$

The integrating factor for the terms in the components of the portfolio is the discount factor e^{-rt}. Multiplying both sides by e^{-rt} and factorizing

$$\frac{d}{dt}e^{-rt}\left(S_t h_t - V_t\right) - e^{-rt}(S_t + \sigma^2 S_t^2 \Gamma_{t,T})\frac{dh_t}{dt} = (\mu + \tfrac{1}{2}\sigma^2 - r)e^{-rt} S_t h_t$$

where each term is now in present values and the term in the portfolio contents is an exact derivative which can be integrated. Integrating from the time the portfolio is established at the backward variable (time zero) to the time the option expires at the forward variable (time *T*)

$$e^{-rT}\left(S_T h_T - V_T\right) - \int_0^T e^{-rt}(S_t + \sigma^2 S_t^2 \Gamma_{t,T})dh_t - \left(S_0 h_0 - V_0\right) = (\mu + \tfrac{1}{2}\sigma^2 - r)\int_0^{T-dt} e^{-rt} S_t h_t dt$$

or in terms of the portfolio variable $\Pi_t = S_t h_t - V_t$

$$e^{-rT}\Pi_T - \int_0^T e^{-rt}(S_t + \sigma^2 S_t^2 \Gamma_t)dh_t - \Pi_0 = (\mu + \tfrac{1}{2}\sigma^2 - r)\int_0^T e^{-rt} S_t h_t dt$$

This is the net present value (NPV) of the replicating portfolio cashflows. The return on the market asset underlying the derivative is $\mu + \frac{1}{2}\sigma^2$. Given that this is larger than *r*, the cashflows for a call ($h_t \geq 0$) results in a profit (and a loss for a put option for which trades short the market asset $h_t \leq 0$), i.e., the present value of **(1)** the closing out proceeds less **(2)** the cost of the delta updates and **(3)** the cost of establishing the hedge portfolio is

$$(\mu + \tfrac{1}{2}\sigma^2 - r)\int_0^T e^{-rt} S_t h_t dt$$

which is the market returns earned, net of cash, on the continually rebalanced position in the market asset in the portfolio throughout the life of the option and replicating portfolio

closing out proceeds - initial cost - rebalancing costs
=
market return on asset holdings

Whenever the return on the underlying asset exceeds the cash rate $\mu + \frac{1}{2}\sigma^2 > r$, the replicating portfolio of the accurate premium is profitable for a call option and loss making for a put option (since $h_t \geq 0$ with a long positions in *S* for a call and $h_t \leq 0$ with a short positions in *S* for a put option). This is consistent with taking a long or short position and pecuniary interest in the economic activity of the market asset for a deferred long or short spot trade, respectively, in the market asset.

The term for the return on assets is the revenue earned, in return for the flatlined option, for the financial service of making a market in the derivative and **is** the **economic value of the derivative earned for taking up the pecuniary interest in the economic activity of the market asset for the spot transaction for which the derivative is deferring.**

The extent of this profit or loss, like the binomial replicating portfolio, is the extent to which the earnings on the market asset exceed the cash rate. When the return on the underlying market asset $\mu + \frac{1}{2}\sigma^2$ is set equal to the cash rate *r*, by debasing it [or the premium] with slack for example, the market return net of cash $\mu + \frac{1}{2}\sigma^2 - r$ vanishes leaving debased portfolio cashflows and the result of return-neutral valuation, namely, all cashflows void of market return and derivatives [as deferred transactions] that have no economic value and which are consequently redundant and worthless securities. This also implies that all market variables upon which derivatives are written also have no [future] economic value. As was seen for the binomial walk and is now shown for exponential Brownian motion, the market returns on the traded market asset could hypothetically be moved into the initial premium resulting in a premium very near the debased premium which is priced accurately for, and only for, the demeaned asset *Y* by debasing returns with the slack variable Δ. It is then proved that the exact statistical bias, as the difference between the accurate premium and the debased premium, is alternatively expressed as the expected present value of the returns given up in order to subsidize the mispricing in the debased price by trading the market asset with deltas of the debased premium [which are smaller {with market volatility} than the deltas of the accurate premium and hence earn slightly less overall return].

If it was possible to neutralize every single market fluctuation and maintain a continuous stream of delta rebalances to maintain parity between the option delta and the offsetting replicating delta by rebalancing infinitely often [to neutralize every single market fluctuation], then taking profits that are earned from a convex delta-

neutral replicating portfolio would be a sensible thing to do for a market maker. There is no sensible reason for trying to bar someone from doing so. Does someone who can buy bubble gum for 5cents wholesale and sell it for 10cents retail become infinitely rich? No. Should they be barred from doing so? No, that would be **communism**. To say that anyone can become infinitely rich by earning an arbitrary spread, and hence should be banned from doing so, is absurd and contrary to all sensible notions of **free and efficient markets**. In this case, the spread is revenue earned for the service of retailing bubble gum, and the 5cents margin per bubble gum is the economic value of the service [by the value added method of calculating GDP]. In the case of derivatives, it is revenue earned for the financial service provided by making a market in a derivative and, if the derivative is replicated with an offsetting position in the market asset, taking up the pecuniary interest in the economic activity of the market asset for the deferred transaction of the derivative. In which case, the revenue earned is not infinite, but instead in direct proportion with the derivative's delta. It is hence **not a super-normal profit or usury**, but a **fair and appropriate margin** for the service provided by making a market in the derivative. For a future, the margin is known in advance. For an option, the margin is stochastic [because of a stochastic delta dependent on the forward transition probability via the CDF or complementary CDF]. In the terminology of the **Margin on Services Method** of allocating earned profit [practiced by Australian and British life insurance actuaries], the delta is the profit carrier. If someone can immunize and profit, and contribute to the goodness of markets by making a market in the derivative and taking the right side of the spot markets, then they should be praised. The ability to do so should not [attempted to] be stamped out, as per **return-neutral valuation and communism**. Markets are all about making the most of, and taking advantage of, opportunities independently of the specifics of the market or whether they are markets for spot or deferred transactions, in this case, of financial securities.

B. THE DEBASED REPLICATING PORTFOLIO OF RETURN-NEUTRAL VALUATION AND THE DEBASED PREMIUM

Introduction

The debased portfolio in continuous time occurs by the same constraint as the binomial walk, whereby the portfolio return is fixed flat at *r*, $d\Pi_t = r\Pi_t dt$, which surreptitiously forces $\mu + \frac{1}{2}\sigma^2 = r$ by varying the parameter μ with slack Δ , i.e., putting $\mu' = \mu - \Delta$ so that $\mu' + \frac{1}{2}\sigma^2 = r$ and $(\mu - \Delta) + \frac{1}{2}\sigma^2 = r$ for which solving for Δ results in the slack variable $\Delta = \mu + \frac{1}{2}\sigma^2 - r$ which demeans the market Brownian motion to *Y* and debases the portfolio for a debased premium. The same slack then reappears with volatility skew when the debased price is set to market prices, so that the slack introduced via the mean μ with $d\Pi_t = r\Pi_t dt$ is then backed out again via the volatility σ causing the skew in σ^{IV} vs *k* since the bias of the debased price is a function of *k* [as well as term-to-maturity], exactly as for the systematic skew observed in implied volatility. The slack variable introduced via the mean $\Delta = \mu + \frac{1}{2}\sigma^2 - r$, as additive slack in the exponential drift, is then numerical when backed out via volatility. The backed out numerical skewed volatility is then $\sigma^{IV} = c\sigma$ where $c = c(\Delta, k)$ is proportional slack and a function of Δ introduced via the mean.

Debasement of the Option Portfolio; the Demeaned Asset Y, and the Option Priced for the Demeaned Asset Y

An option portfolio, the statistically accurate option premium, and the resulting PDE get debased by holding the portfolio total rate of change fixed flat at *r*, $d\Pi = r\Pi dt$, which inadvertently kills the convex market flux of the portfolio return that is otherwise earned from trading the replicating market asset *S*. The result is a debased option premium that is accurate for, and only for, the demeaned asset *Y* and hence statistically inaccurate [statistically biased] for the market asset to deliver *S*. Out of the context of a replicating portfolio which flatlines the option payoff profile, the debased price is outright wrong. For example, if you want the option payoff profile and therefore must not replicate and kill it, then there is no way of earning back the statistical mispricing. In the context of a replicating portfolio, all the debased price does is void all cashflows of market return and turns options into redundant securities of no economic value. The debased premium voids all cashflows of market return by sinking the market return of the traded asset with the mispricing in the premium. The mispriced option is then subsidized by trading returns earned from flatlining it with the offsetting replicating asset.

In contrast to the binomial context where there are several alternative methods to debase the option portfolio of market return with slack, in continuous time there is only one method to debase an option and an option portfolio and that is to deduct slack Δ directly from the exponential drift μ. Unlike, for example, the binomial walk

where step ratios are varied to debase the asset, debasing by taking slack Δ from return μ does not change the diffusion sample space, which remains the positive part of the real line simply because the sample space is dense and continuous, and instead only changes the mean of the probability distribution over the sample space and hence changes the allocation of probability over the unchanged sample space [similarly to varying p to debase the asset in the binomial walk]. A location-scale change of any variable creates another variable that has correlation one with the original variable. So, unlike the binomial walk where p is varied to debase the asset, which hypothetically changes the probability distribution whilst leaving the sample space unchanged, the two variables for Brownian motion remain correlated [those for the binomial walk with p varied do not] so that the demeaned asset moves with the market asset but does so without drift net of cash. Therefore, debasing by taking slack Δ from μ then results in another **distinct** variable which is the original market asset with a sink, a sink that kills the market return whilst leaving the variance and market risk completely unchanged. This is then a risky market asset that has no return; this variable is a demeaned market variable Y, and is the market asset with a sink in the slack variable placed on the diffusion variable of the market asset $Y(T) = S(T)\mathrm{e}^{-\Delta T} = S(T)\mathrm{e}^{-(\mu+\frac{1}{2}\sigma^2-r)T}$ which, on logarithms is $\log Y(T) = \log S(T) - \Delta T$ for $\Delta = \mu + \frac{1}{2}\sigma^2 - r$. Similarly to the binomial walk, and for reasons that should be pretty blatantly clear, this notional asset Y is referred to as either **(1)** the demeaned asset or **(2)** the debased asset or **(3)** the oxymoron risky bank account or **(4)** zero returning asset or **(5)** a redundant asset or **(6)** the return-castrated asset or **(6)** the return-neutered asset or **(7)** a defunct asset or **(8)** a stochastically dominated asset. The distribution for this demeaned asset is referred to by risk-neutral methodology as the risk-neutral measure which is a misnomer, since it is return-neutral not risk-neutral. The method of debasing the market asset for continuous exponential Brownian motion is exactly analogous to the binomial walk whereby slack is proportionately deducted directly from both return ratios, the upstep $u' = u/\Delta$ and downstep $d' = d/\Delta$ [making upsteps earn less and downstep incur more, rather than making upsteps less likely and downsteps more likely], with $\Delta = (puv + qdv)$ and results in the distinct variable Y for the demeaned binomial market asset $Y_n = S_n/(puv + qdv)^n$, i.e., S with a sink

$$Y_n = S_0(u')^X(d')^{n-X} = S_0(u/\Delta)^X(d/\Delta)^{n-X} = S_0u^Xd^{n-X}/\Delta^n = S_n/\Delta^n = S_n/(puv + qdv)^n$$

On logarithms this is simply a linear shift $\log Y_n = \log S_n - n\log(puv + qdv)$ exactly analogously to the demeaned asset of Brownian motion considered here and therefore depends on the same basis function as the asset for which is not debased, and kills all market return within the portfolio cashflows. Y and S are necessarily distinct variables but, since Y is just a shifted version of S on logarithms, both remain dependent upon the same stochastic basis function [unlike that for the binomial walk where relative upstep frequency p is varied which implies a different basis function because of changed relative frequency]. Note that it is not possible, in any circumstance, for one variable to have two different probability distributions at the same time; an asset cannot be both S and Y at the same time; it must be one or the other and the premium must be calculated on the transition distribution of the market asset S upon which the option is written, and for which it expires, otherwise the option is mispriced. Hence, it is the correct assertion of this thesis that there is only one distribution upon which to calculate an appropriate accurate premium for an option on the market asset, and that distribution is [not surprisingly] the transition distribution of the market asset S not that which has been demeaned and debased of return μ.

The transition distribution of the underlying market asset S of the financial derivative, as a deferred spot transaction of the market asset, is

$$S(T) \sim \log N\left(\log S_0 + \mu(T - t_0),\ \sigma^2(T - t_0)\right)$$

conditional on the spot price S_0 at time t_0. **This is the distribution, and the only distribution, for which an accurate premium for a financial derivative on S can be calculated**. The correct distribution of the market asset S is then primed with slack, and hence debased, to the oxymoron risky bank account $S' = Y$ that returns nothing net of cash by varying the parameter μ to $\mu' = r - \frac{1}{2}\sigma^2$ [but retains full market risk σ, i.e., the asset is demeaned without changing the exponential variance in any way] by killing the market return μ with the slack variable $\Delta = \mu + \frac{1}{2}\sigma^2 - r$

$$\mu' = \mu - \Delta = \mu - (\mu + \tfrac{1}{2}\sigma^2 - r) = r - \tfrac{1}{2}\sigma^2$$

i.e.,

$$S(T)' = Y(T) \sim \log N\left(\log S_0 + \mu'(T-t_0),\ \sigma^2(T-t_0)\right)$$
$$\sim \log N\left(\log S_0 + (\mu-\Delta)(T-t_0),\ \sigma^2(T-t_0)\right)$$
$$\sim \log N\left(\log S_0 + (r-\tfrac{1}{2}\sigma^2)(T-t_0),\ \sigma^2(T-t_0)\right)$$

conditional on the spot price, $S(t_0)' = Y(t_0) = S_0$ which can, by conditioning, be anything you like since it is a Markov process and hence has no memory; here it remains conditioned on a notional spot value of the **market asset** since the primed variable $S' = Y$ is just that of the original variable [at the time the option is priced and the drift is notionally killed with slack for the pricing] with debased drift within the transition distribution. Noting that if *Y* and *S* are in parity at the backward variables, by conditioning (with $Y(t_0) = Y_0 = S_0$), then the two variables diverge at the very next instant because ***S* grows more quickly [at market return] that $S' = Y$ [at cash]**; they only return again to parity by again conditioning. The distribution of the debased asset is the demeaned transition distribution conditional on the spot price of the market asset $Y(t_0) = Y_0 = S_0$. Written in full conditional notation this is

$$Y(T)\,|\,Y(t_0) = S_0$$
$$\sim \log N\left(\log S_0 + (r-\tfrac{1}{2}\sigma^2)(T-t_0),\ \sigma^2(T-t_0)\right)$$

for which the total return for a forward transition $T-t_0$ to option expiry is $(r-\frac{1}{2}\sigma^2)(T-t_0)$, which is less than a bank account, rather than $\mu(T-t_0)$ of the market asset. At the same time, the total risk for the transition is $\sigma^2(T-t_0)$ which is exactly the same total risk of the market asset. Hence, this hypothetical asset for which a debased option is priced is an oxymoron in all other financial wisdom and theory. Why would an investor invest in a market asset that has no return when he can invest in one that does have return for exactly the same risk? Alternatively, why would an investor invest in a bank account with risk when he could just put his money in one that does not have risk? What sort of a dumb idiot would the investor have to be? For utility theory, which says that investors always prefer more return to less return [for a given level of risk] and that investors always prefer less risk to more risk [for a given level of return], the demeaned asset *Y* will always be shunned for either a bank account or the market asset *S*. In esoteric finance theory, *Y* is stochastically dominated by *S* at all orders. Consequently, *Y* is also referred to as the oxymoron risky bank account and is a redundant security just like options and other derivatives which are similarly debased and priced for this redundant security. The distribution of the oxymoron risky bank account is called the risk-neutral measure by risk-neutral valuation, which is a complete misnomer [much more appropriately called the return-neutered measure], and the act of neutering a market variable [debasing it of return] is referred to by return-neutered valuation as a *radon-mickey-wang-derivative*[5] [although not by this thesis]. Here is the proper theory [original to this thesis];

In depth theory from BOOK 1, FOUNDATIONS I

The diffusion variable of the market asset *S* at time *T* is *S*(*T*). The diffusion variable of the market asset at time *t* is *S*(*t*). The forward transition of the diffusion variable of the market asset *S* from time *t* to time *T* is the diffusion variable at time *T*, which is *S*(*T*), conditional on [given the value of] the diffusion variable at time *t*, which is *S*(*t*), having the spot value S_t, i.e., $S(t) = S_t$. This constant, S_t, is the backward diffusion variable of the market asset *S*. The conditional variable of the forward transition of the market asset *S* is then written with conditional [bar] notation as $S(T)\,|\,S(t) = S_t$. The distribution of this conditional variable is then the forward transition distribution of the market asset, which is log-normal with exponential drift μ and exponential volatility σ^2

$$S(T)\,|\,S(t) = S_t$$
$$\sim \log N\left(\log S_t + \mu(T-t),\ \sigma^2(T-t)\right)$$

The diffusion variable of the demeaned asset *Y* at time *T* is *Y*(*T*). The diffusion variable of the demeaned asset at time *t* is *Y*(*t*). The forward transition of the diffusion variable of the demeaned asset *Y* from time *t* to time *T* is the diffusion variable at time *T*, which is *Y*(*T*), conditional on [given the value of] the diffusion variable at time *t*, which is *Y*(*t*), having the notional spot value Y_t, i.e., $Y(t) = Y_t$. This constant, Y_t, is the backward diffusion

[5] less facetiously as a Radon-Nickodyme derivative which is measure gibberish; the scale factor of a transformation of a random variable is called a Jacobian. The Jacobian of the transformation to demean the market asset is $dY/dS = \exp(-\Delta T)$, i.e., the exponential sink in the slack variable.

variable of the demeaned asset *Y*. The conditional variable of the forward transition of the demeaned asset *Y* is then written with conditional notation as $Y(T)\,|\,Y(t)=Y_t$. The distribution of this conditional variable is then the forward transition distribution of the demeaned asset, which is log-normal with exponential drift $r-\frac{1}{2}\sigma^2$ and exponential volatility σ^2

$$Y(T)\,|\,Y(t)=Y_t \sim \log N\left(\log Y_t+(r-\tfrac{1}{2}\sigma^2)(T-t),\ \sigma^2(T-t)\right)$$

The debased replicating portfolio surreptitiously places a sink $e^{-\Delta(T-t)}$ in the slack variable Δ on the backward variable of the market asset *S* for the forward transition to option maturity at time *T*, i.e., replaces S_t with $S_t e^{-\Delta(T-t)}$ in the forward transition distribution of the market asset *S*, which is then $S(T)\,|\,S(t)=S_t e^{-\Delta(T-t)}$, for the pricing of the option. This debases the term $\Delta S_t h_t$ from the flux of the accurate PDE leaving it debased for a debased premium. The distribution of the market asset with the sink placed on the backward variable is

$$\begin{aligned}S(T)\,|\,S(t)=S_t e^{-\Delta(T-t)} &\sim \log N\left(\log(S_t e^{-\Delta(T-t)})+\mu(T-t),\ \sigma^2(T-t)\right)\\ &\sim \log N\left(\log S_t+(\mu-\Delta)(T-t),\ \sigma^2(T-t)\right)\\ &\sim \log N\left(\log S_t+(r-\tfrac{1}{2}\sigma^2)(T-t),\ \sigma^2(T-t)\right)\end{aligned}$$

which is exactly the demeaned asset resulting from the market asset being debased with the slack variable Δ, i.e., that obtained simply by killing μ with Δ in the transition distribution of *S*. This is now shown to be the distribution of the demeaned asset, as conditional on the spot value of the market asset at time *t*. The conditional variable $S(T)\,|\,S(t)=S_t e^{-\Delta(T-t)}$ has the same distribution as $S(T)\,|\,S(t)e^{\Delta(T-t)}=S_t$. Because the diffusion is both **proportional** and **time-homogenous,** this is equivalent in distribution to $S(T)e^{-\Delta T}\,|\,S(t)e^{-\Delta t}=S_t$. The demeaned asset *Y* is the market asset *S* with a sink. At time *T*, this is defined as $Y(T)=S(T)e^{-\Delta T}$ and at time *t* it is defined as $Y(t)=S(t)e^{-\Delta t}$. Therefore the conditional variable $S(T)e^{-\Delta T}\,|\,S(t)e^{-\Delta t}=S_t$ can be written as $Y(T)\,|\,Y(t)=S_t$ which is the conditional distribution of the demeaned asset *Y*, conditional on the spot value of the market asset *S* and is hence log-normal with exponential drift $r-\frac{1}{2}\sigma^2$ and exponential volatility σ^2

$$Y(T)\,|\,Y(t)=S_t \sim \log N\left(\log S_t+(r-\tfrac{1}{2}\sigma^2)(T-t),\ \sigma^2(T-t)\right)$$

This is the distribution of the oxymoron risky bank account *Y*, conditional on the spot value of the market asset, for which debased option premiums are calculated and is hence wrong for derivatives written on market assets and results in wrong premiums at the debased prices which are inaccurate and statistically biased. This distribution is remeaned again, to that of the market asset, by placing a source for the forward transition $e^{\Delta(T-t)}$ either on the forward variable or on the backward variable. This theory is utilized again when it comes to remeaning the debased premiums in order to get an accurate price consistent with prices observed in the option markets [which are not debased prices], as a superior alternative to skewing volatility which is suboptimal and not minimum variance.

End of in depth

The transition density of the oxymoron bank account *Y* is similarly that for *S* with the exponential drift μ varied for, and hence debased with, the slack variable Δ so that μ is replaced by μ' where $\mu'=\mu-\Delta=\mu-(\mu+\frac{1}{2}\sigma^2-r)=r-\frac{1}{2}\sigma^2$ for

$$f_{Y(T)|Y(t)}(x\,|\,S_0)=\frac{1}{x\sigma\sqrt{T-t_0}\sqrt{2\pi}}e^{-\frac{1}{2}\frac{[\log x-\log S_0-(r-\frac{1}{2}\sigma^2)(T-t_0)]^2}{\sigma^2(T-t_0)}}$$

The total variance in the density for the forward transition, and hence total market risk, remains unchanged at $\sigma^2(T-t)$. The diffusion equation and backward equation of the demeaned asset *Y* are

$$\frac{dY_t}{dt} = rY_t + \sigma Y_t \frac{dB_t}{dt} \qquad \text{and} \qquad -\frac{\partial}{\partial t} f_{Y(T)|Y(t)}(x \mid Y_t) = \tfrac{1}{2}\sigma^2 Y_t^2 \frac{\partial^2}{\partial Y_t^2} f_{Y(T)|Y(t)}(x \mid Y_t) + rY_t \frac{\partial}{\partial Y_t} f_{Y(T)|Y(t)}(x \mid Y_t)$$

which, for a debased premium and option portfolio, are solved subject to the initial condition that the demeaned asset and the market asset are in parity, i.e., that the spot value of the notional demeaned asset coincides with the spot value of the market asset at the time the option is priced [i.e., at the backward variables]. The recursion for the density *f* of *Y* is

$$f_{Y(T)|Y(t_0)}(x \mid S_0) = \int_0^\infty f_{X'}(y)\, f_{Y(T)|Y(t_0+dt)}(x \mid Y_0 \times y) dy$$

where the basis components X' are primed with the slack variable and hence the exponential mean is debased from μ to $r - \frac{1}{2}\sigma^2$ for the infinitesimal changes in the forward transition distribution *f* as each log-normal basis component $X' \sim \log N(r - \frac{1}{2}\sigma^2, \sigma^2)$ is deconvolved from the total forward transition distribution within the transition density.

The debased premium $V^{debased}$ is the premium priced for the demeaned asset *Y* rather than the market asset *S* of the derivative. The difference between these two premiums is the statistical bias and inaccuracy of the debased price. In the context of replicating the option with an offsetting asset, which flatlines the option, the market returns earned on trading the offsetting asset attempt to earn back the mispricing by subsidizing the mispricing with market returns. All functions of the statistically accurate formulae for options and option replicating portfolios get debased by varying the exponential drift μ (the market return) with the slack variable Δ, i.e., replacing the exponential drift μ, wherever it occurs, with the debased drift $\mu' = \mu - \Delta = r - \frac{1}{2}\sigma^2$ so that all functions are void of market return net of cash.

Hence, the debased option premium for exercise price *k* maturing in $T - t_0$ periods is that for the statistically accurate premium for the market asset to deliver but with the mean debased with the slack variable and hence accurate for the demeaned asset *Y* rather than the market asset *S*. For a call option, debasing (priming) the accurate premium for *S* with the slack variable Δ for the debased premium gives

$$V^{debased}(k,T;S_0,t_0) = S_0 \overline{F}_U\left(u_k' - \sigma\sqrt{T-t_0}\right) - ke^{-r(T-t_0)}\overline{F}_U\left(u_k'\right)$$

where $\overline{F}_U(\cdot)$ is the complementary CDF of a unit normal $U \sim N(0,1)$ and u_k' is a normal pivot for *U* and the standardized exercise price for the logarithm of the demeaned asset *Y* [and hence primed with slack] for a forward transition of length $T - t_0$

$$u_k' = \frac{\log k - \log S_0 - \mu'(T-t_0)}{\sigma\sqrt{T-t_0}} = \frac{\log k - \log S_0 - (r - \frac{1}{2}\sigma^2)(T-t_0)}{\sigma\sqrt{T-t_0}}$$

This is the Black-Scholes return-neutered formula which is the debased price of a call option derivative. The debased future value of the spot asset is $e^{(\mu' + \frac{1}{2}\sigma^2 - r)(T-t_0)} = 1$ so that the debased premium has the same structural form as the accurate premium but with a null future value; no return on the asset in the coefficient and no drift in excess of cash in the standardized exercise price u_k'. Hence, this is the formula for an asset with a killed future value and a probability junked with statistical bias. For market return in excess of cash, the **debased call option premium** is **statistically biased downwards**, i.e., statistically **underprices the call option** to buy the market asset [as seen empirically, and hence admitted, by Black and Scholes in 1973]. The term $\overline{F}_U\left(u_k'\right)$ in this formula is the probability that a call option on the demeaned asset *Y* will expire inthemoney, that for which it is priced, not the probability that the option on the actual underlying market asset *S* will expire inthemoney for which has the proper rate of exponential drift in the standardized exercise price. The difference between these two probabilities is the statistical bias of that junked with slack for the debased premium; it is accurate only for *Y*. This probability for the call option [upside], like the premium and for the same reasons, is statistically biased downwards and hence under-estimates the probability that the call option on *S* will expire inthemoney, i.e., underestimates the probability, that *S* will be above the exercise price upon expiry of the deferral period of the derivative. This [the wrong expiry probability] evidences clearly that the debased call option premium underestimates the price of the call option. Note that **return-neutered valuation and Black and Scholes actively excluded any other premium**, other than the debased premium, including that of the accurate premium [by discrediting and being bastards to anyone who wanted to be sensible]. This is why this book of this thesis, BOOK 2, has been written, namely, to make the case [which is unequivocal] for accurate premiums and for the economic val-

ue of derivatives to be fully recognized, as securities that are not redundant and worthless [which they are under return-neutered valuation] and put an end to ignorant assholes being bastards to sensible people like me.

For a put option, the debased premium from debasing (priming) the accurate premium with the slack variable Δ is

$$V^{debased}(k,T;S_0,t_0) = ke^{-r(T-t_0)}F_U\left(u_k'\right) - S_0F_U\left(u_k' - \sigma\sqrt{T-t_0}\right)$$

where $F_U(\cdot)$ is the CDF of a unit normal $U \sim N(0,1)$ and u_k' is similarly that of the debased call option premium. This is the Black-Scholes return-neutered formula which is the debased price of a put option derivative. Similarly to the debased call option premium, **(1)** the future value is debased in the coefficient of the spot price and **(2)** the nominalized exercise price has drift that does not exceed cash; hence the expiry probabilities are junked with statistical bias. For market return in excess of cash, the **debased put option premium** is **statistically biased upwards**, i.e., statistically **overprices the put option** to sell the market asset. The term $F_U\left(u_k'\right)$ in the formula is the probability that the put option on the demeaned asset *Y* will expire inthemoney, not the probability that the option upon the underlying market asset *S* will expire inthemoney. Similarly to the debased call option, the difference between this probability and the accurate probability is the statistical bias of the expiry probability of that junked with slack. This probability for the put option [downside], like the premium and for the same reasons, is statistically biased upwards and hence over-estimates the probability that the put option on *S* with expire inthemoney, i.e., overestimates the probability that *S* will be below the exercise price upon expiry of the deferral period of the derivative. This [the wrong expiry probability] then evidences clearly that the debased put option premium overestimates the price of the put option.

These are premiums priced for the demeaned asset *Y* so are statistical expectations and integrals with respect to the debased density which is, of course, that of the accurate density but with the drift varied for slack and hence with debased return $\mu' = \mu - \Delta = \mu - (\mu + \frac{1}{2}\sigma^2 - r) = r - \frac{1}{2}\sigma^2$ but with full market risk σ of the market asset. Note that if the debased option pricing formula where prices for a situation that was risk free, then the formula would not contain normal distribution functions, which of course it does. It necessarily contains σ otherwise it would not be a premium for Brownian motion; what it does not contain is market return μ. Hence, this thesis also calls the **debased premium** the **return-free premium**, because it is **void of market return** [yet is priced for **full market risk**]. This thesis also calls the **debased premium** the **variance-only premium**, because it is the premium for **full market risk** [and **no market return**].

The expectations with respect to the density of the demeaned asset *Y* integrate for premiums and deltas similarly as for that with the correct transition distribution. These integrations are omitted since you get them simply by debasing by replacing μ with μ'. Note that put-call parity, and the price of the underlying derivative for the future [the derivative without the option to walk away from the future spot trade], also gets debased. The mean of the demeaned asset is then that of the market asset with all return above cash, the market future value, killed by the slack variable Δ. It is therefore put-call parity for assets that have no future economic value, in particular, assets whose future economic activity [over the deferral period of the derivative] has no value [or at least are priced as such]. The present value of the mean of the forward transition to option expiry is then the spot price, and **debased put-call parity** is then

$$\begin{aligned} c_{t,T}^{debased} - p_{t,T}^{debased} &= S_0 e^{(\mu' + \frac{1}{2}\sigma^2 - r)(T-t)} - ke^{-r(T-t)} \\ &= S_0 - ke^{-r(T-t)} \end{aligned}$$

which says that all future value of any economic asset is completely null in the context of the spot financial markets and deferring transactions of these markets [which define financial derivatives]. Not surprisingly, **debased put-call parity** never equates to put-call parity for market prices for options. **Debased put-call parity** does not hold for market prices since market prices are not debased prices, and never have been, as consistently observed since 1973 (all that time ago). Getting the debased put option premium from the debased call option premium

$$\begin{aligned} p_{t,T}^{debased} &= c_{t,T}^{debased} - S_0 + ke^{-r(T-t)} \\ &= S_0\overline{F}_U\left(u_k' - \sigma\sqrt{T-t_0}\right) - ke^{-r(T-t_0)}\overline{F}_U\left(u_k'\right) - S_0 + ke^{-r(T-t_0)} \\ &= S_0\left(\overline{F}_U\left(u_k' - \sigma\sqrt{T-t_0}\right) - 1\right) - ke^{-r(T-t_0)}\left(\overline{F}_U\left(u_k'\right) - 1\right) \\ &= ke^{-r(T-t_0)}F_U\left(u_k'\right) - S_0F_U\left(u_k' - \sigma\sqrt{T-t_0}\right) \end{aligned}$$

as before. Differentiating the option premium with respect to the backward variable for the portfolio delta

$$h^{debased}(k,S_0;T,t_0)=\frac{\partial}{\partial S_0}V^{debased}(k,S_0;T,t_0)$$

gives

$$\begin{aligned}h^{debased}(k,S_0;T,t_0)&=\frac{\partial}{\partial s_0}\left(S_0\overline{F}_U\left(u_k'-\sigma\sqrt{T-t_0}\right)-ke^{-r(T-t_0)}\overline{F}_U\left(u_k'\right)\right)\\&=\overline{F}_U\left(u_k'-\sigma\sqrt{T-t_0}\right)+\left(S_0\frac{\partial}{\partial S_0}\int_{u_k'-\sigma\sqrt{T-t_0}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx-ke^{-r(T-t_0)}\frac{\partial}{\partial S_0}\int_{u_k'}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx\right)\\&=\overline{F}_U\left(u_k'-\sigma\sqrt{T-t_0}\right)+\left(-S_0\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}[u_k'-\sigma\sqrt{T-t_0}]^2}+ke^{-r(T-t_0)}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}[u_k']^2}\right)\frac{\partial u_k'}{\partial s_0}\\&=\overline{F}_U\left(u_k'-\sigma\sqrt{T-t_0}\right)\end{aligned}$$

for the delta of the debased call option premium and

$$\begin{aligned}h^{debased}(k,S_0;T,t_0)&=\frac{\partial}{\partial S_0}\left(ke^{-r(T-t_0)}F_U\left(u_k\right)-S_0F_U\left(u_k'-\sigma\sqrt{T-t_0}\right)\right)\\&=-F_U\left(u_k'-\sigma\sqrt{T-t_0}\right)+\left(ke^{-r(T-t_0)}\frac{\partial}{\partial S_0}\int_{-\infty}^{u_k'}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx-S_0\frac{\partial}{\partial S_0}\int_{-\infty}^{u_k'-\sigma\sqrt{T-t_0}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx\right)\\&=-F_U\left(u_k'-\sigma\sqrt{T-t_0}\right)+\left(ke^{-r(T-t_0)}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}[u_k']^2}-S_0\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}[u_k'-\sigma\sqrt{T-t_0}]^2}\right)\frac{\partial u_k'}{\partial S_0}\\&=-F_U\left(u_k'-\sigma\sqrt{T-t_0}\right)\end{aligned}$$

for the delta of the debased put option premium These deltas are also obtainable directly from those of the accurate premiums by debasing the mean with the slack variable, by replacing μ with $\mu'=\mu-\Delta$ in the deltas of the accurate premiums, as are all other functions. Alternatively from differentiating debased put-call parity

$$\frac{\partial}{\partial S_0}\left(c_{t,T}^{debased}-p_{t,T}^{debased}\right)=\frac{\partial}{\partial S_0}\left(S_0-ke^{-r(T-t)}\right)$$

i.e.,

$$\frac{\partial c_{t,T}^{debased}}{\partial S_0}-\frac{\partial p_{t,T}^{debased}}{\partial S_0}=1$$

so

$$\frac{\partial p_{t,T}^{debased}}{\partial S_0}=\frac{\partial c_{t,T}^{debased}}{\partial S_0}-1=\overline{F}_U\left(u_k'-\sigma\sqrt{T-t_0}\right)-1=-F_U\left(u_k'-\sigma\sqrt{T-t_0}\right)$$

as before. For the accurate premium, the portfolio deltas carry the return for the market asset and the economic value of the derivative for a transition length of *dt* . These deltas do not contain return [if calculated with market volatility].

The backwards deconvolution recurrence for the debased option premium $V^{debased}$

Letting primed *X* be the debased basis function X' . The mean μ of the basis function *X* is debased with Δ to X' which has μ replaced with $\mu'=\mu-\Delta$ where $\Delta=\mu+\frac{1}{2}\sigma^2-r$. The backward deconvolution equation is then

$$V^{debased}(k,T;S_0,t_0)=e^{-rdt}\int_0^{\infty}f_{X'}(y)V^{debased}(k,T;S_0\times y,t_0+dt)dy$$

with the debased basis components primed for slack

$$X_j' \sim \log N\left(\mu' dt, \sigma^2 dt\right)$$
$$\sim \log N\left((r - \tfrac{1}{2}\sigma^2)dt,\ \sigma^2 dt\right)$$

for the convolution integral, where $\mu' = \mu - \Delta = r - \frac{1}{2}\sigma^2$, so the basis density functions are

$$f_{X_j'}(y_j) = \frac{1}{y_j\,\sigma\sqrt{2\pi\,dt}} e^{-\frac{1}{2}\frac{[\log y_j - \mu' dt]^2}{\sigma^2 dt}} = \frac{1}{y_j\,\sigma\sqrt{2\pi\,dt}} e^{-\frac{1}{2}\frac{[\log y_j - (r-\frac{1}{2}\sigma^2)dt]^2}{\sigma^2 dt}}$$

Hence, the debased formula is

$$V^{debased}(k,T;S_0,t_0) = e^{-rdt}\int_0^\infty \frac{1}{y\,\sigma\sqrt{2\pi\,dt}} e^{-\frac{1}{2}\frac{[\log y - (r-\frac{1}{2}\sigma^2)dt]^2}{\sigma^2 dt}} V^{debased}(k,T;S_0\times y, t_0 + dt)dy$$

Derivation of the backwards deconvolution recurrence for the debased option premium $V^{debased}$

Since this recurrence describes how debased basis components X' are deconvolved from the formula for the debased option premium $V^{debased}$ for a forward movement in the backward variables [time decay], and hence convolved into the density of the recursion, it is derived by transforming the recurrence of the density *f* of the demeaned asset *Y* to the debased premium $V^{debased}$ which is then accurate for, and only for, the demeaned asset *Y*. This derivation is four or five lines of working and is almost identical to that of the accurate premium with sole difference that the basis densities within the deconvolution integral are X' rather than *X* so that the recursion is for the distribution of *Y* rather than *S*.

The backwards deconvolution recurrence for the transition density *f* of the demeaned asset *Y* is that with debased basis density functions primed with slack X'

$$f_{Y(T)|Y(t_0)}(x \mid Y_0) = \int_0^\infty f_{X'}(y)\, f_{Y(T)|Y(t_0+dt)}(x \mid Y_0 \times y)dy$$

For the debased price of a **call option**, multiplying both sides by the option intrinsic value at the forward variables $x - k$

$$(x-k)\, f_{Y(T)|Y(t_0)}(x \mid Y_0) = (x-k)\int_0^\infty f_{X'}(y)\, f_{Y(T)|Y(t_0+dt)}(x \mid Y_0 \times y)dy$$

and integrating over the forward variables over the range of moneyness for the excess function in the demeaned asset *Y*

$$\int_k^\infty (x-k)\, f_{Y(T)|Y(t_0)}(x \mid Y_0)dx = \int_k^\infty (x-k)\int_0^\infty f_{X'}(y)\, f_{Y(T)|Y(t_0+dt)}(x \mid Y_0 \times y)dydx$$
$$= \int_0^\infty f_{X'}(y)\left(\int_k^\infty (x-k)\, f_{Y(T)|Y(t_0+dt)}(x \mid Y_0 \times y)dx\right)dy$$

Multiplying both sides by the discount factor $e^{-r(T-t_0)} = e^{-rdt}e^{-r(T-t_0-dt)}$

$$e^{-r(T-t_0)}\int_k^\infty (x-k)\, f_{Y(T)|Y(t_0)}(x \mid Y_0)dx = e^{-r(T-t_0)}\int_k^\infty (x-k)\int_0^\infty f_{X'}(y)\, f_{Y(T)|Y(t_0+dt)}(x \mid Y_0 \times y)dydx$$
$$= e^{-rdt}\int_0^\infty f_{X'}(y)\left(e^{-r(T-t_0-dt)}\int_k^\infty (x-k)\, f_{Y(T)|Y(t_0+dt)}(x \mid Y_0 \times y)dx\right)dy$$

The debased call option premium is defined as $V^{debased}(k,T;Y_0,t_0) = e^{-r(T-t_0)}\int_k^\infty (x-k)\, f_{Y(T)|Y(t_0)}(x \mid Y_0)dx$ hence the recurrence is whence transformed

$$V^{debased}(k,T;Y_0,t_0) = e^{-rdt}\int_0^\infty f_{X'}(y)V^{debased}(k,T;Y_0\times y, t_0 + dt)dy$$

where X' is the debased log-normal basis variable $X_j' \sim \log N(\mu' dt, \sigma^2 dt) \sim \log N\left((r - \frac{1}{2}\sigma^2)dt, \sigma^2 dt\right)$. The mathematics for the transformation of the integral recursion to the debased price of a **put option** is similar [omitted]. The recursion is subject to the option payoff for the inthemoney boundary region of the sample space in the demeaned asset *Y* at option expiry

$$V^{debased}(k,T;Y(T),T)=\begin{cases}Y(T)-k & \text{for } Y(T)>k\\ 0 & \text{for } Y(T)<k\end{cases} \qquad V^{debased}(k,T;Y(T),T)=\begin{cases}0 & \text{for } Y(T)>k\\ Y(T)-k & \text{for } Y(T)<k\end{cases}$$

for a call option and a put option respectively, and is solved for the debased premium subject to the initial condition that the demeaned asset is in parity with the spot price of the market asset, i.e., by conditioning, with the market asset $Y(t_0)=S_0$ at the time of the pricing.

This integral backwards recursion is exactly that for the statistically accurate option premium but with sole difference that the basis function is debased with the slack variable. They are identical in every other way other than the boundary region for which it expires. This expression makes it clear that for each infinitesimal forward step *dt*, further slack is compounded into the forward transition which further exacerbates the statistical bias in the debased price **by a further infinitesimal slack component** Δdt **compounded into the premium**. This recursion for the debased premium is the continuous time equivalent to either $V_0^{debased}=vp'V_u^{debased}+vq'V_d^{debased}$ debased with probability primed with slack or $V_0^{debased}=vpV_{u'}^{debased}+vqV_{d'}^{debased}$ [original to this thesis] with step ratios primed with slack, although most correctly and consistently corresponds to the latter with debased step ratios whereby *Y* and *S* remain correlated [because *Y* is just a location change on logarithms and hence the transformation from *S* is independent of the basis function]. Hence, similarly to the premium for the binomial walk, the backwards recursion can be solved by unfolding it, infinitesimal by infinitesimal, over the backward variables to the forward variables at option maturity $T=t_0+ndt$, to the boundary for the demeaned asset *Y* for which the $n=(T-t_0)/dt$ debased basis variables X' of the forward transition relate. The inaccuracy for this expression solved without remeaning the forward variable for an accurate payoff function is clearly seen with the probabilities which are statistically biased and inaccurate for the probability for an inthemoney exercise for an option on *S*.

Solving the recurrence for the debased option PREMIUM $V^{debased}$

From the recursion for the debased premium, that with debased basis components in the deconvolution integral

$$V^{debased}(k,T;S_0,t_0)=e^{-rdt}\int_0^\infty f_{X'}(y)V^{debased}(k,T;S_0\times y,t_0+dt)dy$$

unfolding to option expiry at steps of *dt*

$$\begin{aligned}V^{debased}(k,T;S_0,t_0)&=e^{-rdt}\int_0^\infty f_{X'}(y_1)V^{debased}(k,T;S_0\times y_1,t_0+dt)dy_1\\&=e^{-rdt}\int_0^\infty f_{X'}(y_2)e^{-rdt}\int_0^\infty f_{X'}(y_1)V^{debased}(k,T;S_0\times y_1\times y_2,t_0+2dt)dy_1dy_2\\&=e^{-rdt}\int_0^\infty f_{X'}(y_3)e^{-rdt}\int_0^\infty f_{X'}(y_2)e^{-rdt}\int_0^\infty f_{X'}(y_1)V^{debased}(k,T;S_0\times y_1\times y_2\times y_3,t_0+3dt)dy_1dy_2dy_3\\&\ldots\\&=e^{-rdt}\int_0^\infty f_{X'}(y_n)\ldots e^{-rdt}\int_0^\infty f_{X'}(y_2)e^{-rdt}\int_0^\infty f_{X'}(y_1)V^{debased}(k,T;S_0\times y_1\times y_2\times\ldots\times y_n,t_0+ndt)dy_1dy_2\ldots dy_n\end{aligned}$$

i.e.,

$$V^{debased}(k,T;S_0,t_0)=e^{-nrdt}\int_0^\infty f_{X'}(y_n)\ldots\int_0^\infty f_{X'}(y_2)\int_0^\infty f_{X'}(y_1)V^{debased}(k,T;S_0\times y_1\times y_2\times\ldots\times y_n,t_0+ndt)dy_1dy_2\ldots dy_n$$

With $n=(T-t_0)/dt$, the option expires in *n* periods of *dt* time units past time t_0 at time $T=t_0+ndt$, this can be written as

$$V^{debased}(k,T;S_0,t_0)=e^{-n(T-t_0)}\int_0^\infty f_{X'}(y_n)\ldots\int_0^\infty f_{X'}(y_2)\int_0^\infty f_{X'}(y_1)V^{debased}(k,T;S_0\times y_1\times y_2\times\ldots\times y_n,T)dy_1dy_2\ldots dy_n$$

The repeated *n*-fold deconvolution of $n=(T-t_0)/dt$ basis terms, each primed with infinitesimal slack Δdt, from the premium *V* is an *n*-fold convolution of the *n* basis terms into the aggregate basis density *f* of the unfolded recursion

$$\begin{aligned}\textstyle\prod_0^n X_j' &\sim \log N\left((r-\tfrac{1}{2}\sigma^2)ndt,\sigma^2ndt\right)\\&\sim \log N\left((r-\tfrac{1}{2}\sigma^2)(T-t_0),\sigma^2(T-t_0)\right)\end{aligned}$$

Hence, the integral for $x = S_0 \times y_1 \times y_2 \times ... \times y_n$ for the *n* basis components convolved onto the backward variable S_0 for the *n*-fold aggregated basis function is

$$Y_T = S_0 \times \prod_0^n X_j' \sim \log N\left(\log S_0 + (r - \tfrac{1}{2}\sigma^2)(T - t_0),\ \sigma^2 (T - t_0)\right)$$

which is the forward transition distribution of *S* debased with slack, i.e., the transition distribution of the oxymoron risky bank account *Y* with **no market return** and **full market risk**. The forward distribution of Y_T, given the spot price S_0 at time t_0 (conditional on $Y(t_0) = S_0$), has density function

$$f_{Y(T)|Y(t)}(x \mid S_0) = \frac{1}{x\sigma\sqrt{ndt}\sqrt{2\pi}} e^{-\frac{1}{2}\frac{[\log x - \log S_0 - (r - \frac{1}{2}\sigma^2)ndt]^2}{\sigma^2 ndt}} = \frac{1}{x\sigma\sqrt{T - t_0}\sqrt{2\pi}} e^{-\frac{1}{2}\frac{[\log x - \log S_0 - (r - \frac{1}{2}\sigma^2)(T - t_0)]^2}{\sigma^2 (T - t_0)}}$$

which is exactly that for the market asset with the exponential drift debased with the slack variable Δ. The level of market risk is unchanged, i.e., the density contains the full amount of market risk $\sigma^2(T - t_0)$ of the market asset *S* for the forward transition of *n*-steps of *dt* to option expiry.

The unfolded recurrence of the debased premium becomes

$$\begin{aligned} V^{debased}(k, T; S_0, t_0) &= e^{-r(T - t_0)} \int_0^\infty f_{S_0 X_1' X_2' ... X_n'}(x) V^{debased}(k, T; x, T) dx \\ &= e^{-r(T - t_0)} \int_0^\infty f_{Y(T)|Y(t)}(x \mid S_0) V^{debased}(k, T; x, T) dx \end{aligned}$$

i.e.,

$$V^{debased}(k, T; S_0, t_0) = e^{-r(T - t_0)} \int_0^\infty \frac{1}{x\sigma\sqrt{T - t_0}\sqrt{2\pi}} e^{-\frac{1}{2}\frac{[\log x - \log S_0 - (r - \frac{1}{2}\sigma^2)(T - t_0)]^2}{\sigma^2 (T - t_0)}} V^{debased}(k, T; x, T) dx$$

which is the definition of the premium and integrates into the debased formula as before. The boundary is that part of the sample space for which an option on the demeaned asset *Y*, rather than the market asset *S*, is inthemoney [hence the wrong probabilities for an inthemoney exercise for an option on *S*]. For a call option, this is

$$V^{debased}(k, T; Y_T, T) = \begin{cases} Y_T - k & if\ Y_T > k \\ 0 & if\ Y_T < k \end{cases}$$

and hence the recursion solves for

$$\begin{aligned} V^{debased}(k, T; S_0, t_0) &= e^{-r(T - t_0)} \int_0^\infty f_{Y(T)|Y(t_0)}(x \mid S_0) V^{debased}(k, T; x, T) dx \\ &= e^{-r(T - t_0)} \int_0^\infty f_{Y(T)|Y(t_0)}(x \mid S_0) \max(Y_T - k, 0 \mid Y_T = x) dx \\ &= e^{-r(T - t_0)} \int_k^\infty (x - k) f_{Y(T)|Y(t_0)}(x \mid S_0) dx \\ &= e^{-r(T - t_0)} \int_k^\infty \frac{(x - k)}{x\sigma\sqrt{T - t_0}\sqrt{2\pi}} e^{-\frac{1}{2}\frac{[\log x - \log S_0 - (r - \frac{1}{2}\sigma^2)(T - t_0)]^2}{\sigma^2 (T - t_0)}} dx \\ &= S_0 \overline{F}_U\left(u_k' - \sigma\sqrt{T - t_0}\right) - k e^{-r(T - t_0)} \overline{F}_U\left(u_k'\right) \end{aligned}$$

For a put option, the boundary is

$$V^{debased}(k, T; Y_T, T) = \begin{cases} 0 & if\ Y_T > k \\ k - Y_T & if\ Y_T < k \end{cases}$$

so the recursion solves for

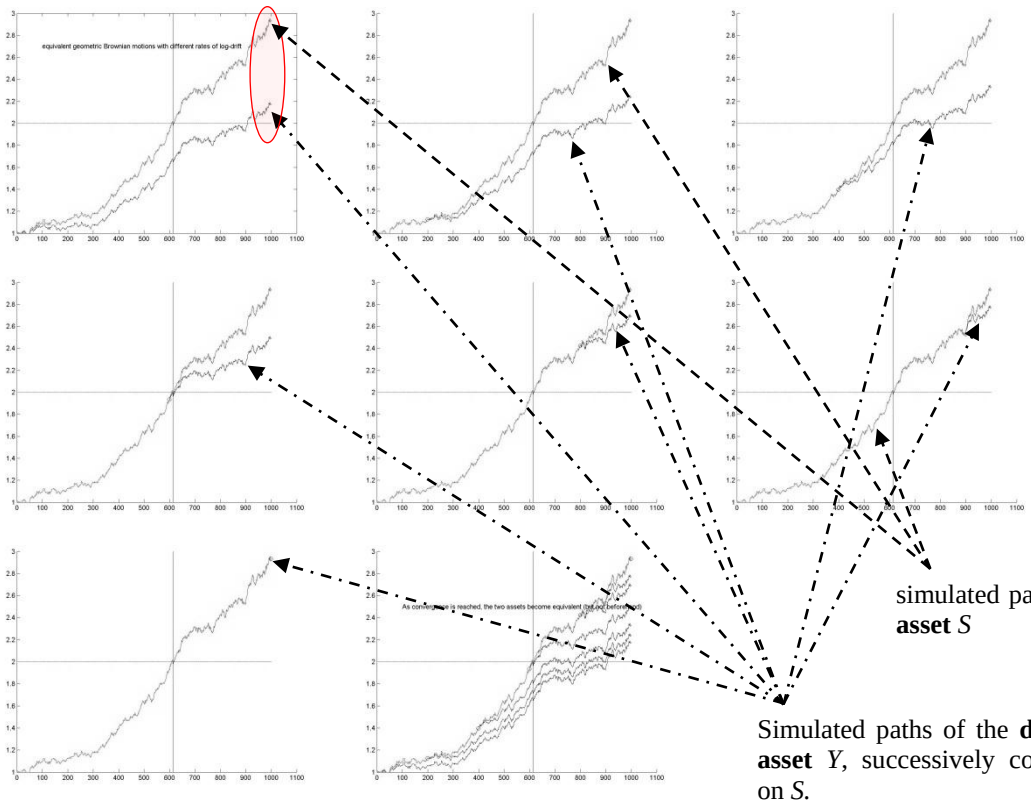
$$
\begin{aligned}
V^{debased}(k,T;S_0,t_0) &= e^{-r(T-t_0)}\int_0^{\infty} f_{Y(T)|Y(t_0)}(x \mid S_0)V^{debased}(k,T;x,T)dx \\
&= e^{-r(T-t_0)}\int_0^{\infty} f_{Y(T)|Y(t_0)}(x \mid S_0)\max(k-Y_T,0 \mid Y_T = x)dx \\
&= e^{-r(T-t_0)}\int_0^{k} (k-x) f_{Y(T)|Y(t_0)}(x \mid S_0)dx \\
&= e^{-r(T-t_0)}\int_0^{k} \frac{(k-x)}{x\sigma\sqrt{T-t_0}\sqrt{2\pi}} e^{-\frac{1}{2}\frac{[\log x-\log S_0-(r-\frac{1}{2}\sigma^2)(T-t_0)]^2}{\sigma^2(T-t_0)}} dx \\
&= ke^{-r(T-t_0)}F_U\left(u_k'\right) - S_0F_U\left(u_k' - \sigma\sqrt{T-t_0}\right)
\end{aligned}
$$

where u_k' is the standardized exercise price and normal pivot for $U \sim N(0,1)$ with respect to a forward transition in the demeaned asset Y

$$u_k' = \frac{\log k - \log S_0 - (r-\frac{1}{2}\sigma^2)(T-t_0)}{\sigma\sqrt{T-t_0}}$$

These are the debased option prices. It is seen that an option for a forward transition of $n=(T-t_0)/dt$ infinitesimal steps from t_0 to T is composed of n components of infinitesimal slack $n\Delta dt$ within the n-fold convolution that composes the forward transition density for which the premium is calculated. Therefore, when implied volatility is backed out from market prices, the skew of the implied volatility depends on the magnitude of the slack variable Δ and time to maturity $ndt = T-t_0$ (as well as the exercise price) since the slack is compounded into the premium with each infinitesimal step that comprises the forward transition. This means that implied volatility of a one month option is not the same as the implied volatility for, say, a two week option for a given exercise price. People that calculate their premiums and deltas with implied volatility, rather than market volatility, and use a flat volatility measure (say one month implied vol) over the replicating period are doing the wrong thing. This results in an over adjustment of the premium at all periods after the portfolio is first established for which the implied volatility applies (say for one month to option expiry with one month implied vol). For example, a two-week option calculated with one-month implied volatility overcompensates with one months worth of bias and compounded slack factored into the debased premium, via volatility, which then has only two weeks worth of bias and compounded slack for a two-week term-to-maturity [because there is also time decay in the total compounded slack in the premium and deltas]. This reverses the direction of the statistical bias. For calls and ordinary markets $\mu+\frac{1}{2}\sigma^2 > r$, which means that implied volatility is greater than market volatility, this means an overpriced call option [which are otherwise underpriced at market volatility at the debased price] and, for puts, this means an underpriced option [which are otherwise overpriced at market volatility at the debased price]. This, then, is a method that is in error if implied volatility is not adjusted at each rebalance for term-to-maturity, which is inconvenient at best, and otherwise impossible [especially considering that it is required for a continuous stream of rebalances all the way up to expiry and for an arbitrary value of the exercise price; no option market has this sort of depth to imply all of these {skewed} volatilities]. Rebalancing infinitely often requires an infinite number of implied volatility measures, each with differing terms to maturity. This is impossible, even so for very few rebalances, and particularly considering that implied volatility is not always defined. The sensible way and most correct method to go about parameterizing the formulae for a replication is to get a value for return, from **market put call parity**, and then use one fixed measure of **market volatility** throughout the replication period. At least this way all formulae are statistically accurate, theoretically, under the given model assumptions.

The following graphics show simulated paths of a proportional [exponential] Brownian motion for a market asset *S*, and also that which has been demeaned *Y*, the resulting variable of *S* with a sink placed on it which kills the market return. The high path is that with market drift (*S*) and the lower paths are those that grow at cash (*Y*), which are successively conditioned at six later points of the backward variable of *S*. The paths are equal only at the point at where they are conditioned; they diverge the very next instant until they are again conditioned [since the growth rates differ].



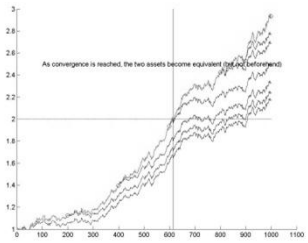
These simulated paths are intentionally high signal-to-noise ratio so that the drift [average exponential gradient] can be visualized clearly. It is seen that immediately after conditioning with $Y_0 = S_0$, the path of *Y* immediately moves out of parity with *S* because of the different rates of drift. The variables are only again brought back into parity by again conditioning, and hence resulting in a new path for any forward transition. In the context of a replicating portfolio trading *S* to replicate a debased option priced for *Y*, conditioning occurs at each point where either an option is priced or a rebalance in *S* takes place. For the above grid, a replicating portfolio would have rebalanced five times between establishing the portfolio [first graphic] and closing out the portfolio [seventh graphic]. The probability that the simulated path of the market asset *S* will be above the exercise price at expiry, calculated at any time prior to expiry say a term-to-maturity of $T - t_0$, is

$$\overline{F}_{N(0,1)}\left(u_k\right) = \overline{F}_{N(0,1)}\left(\frac{\log k - \log S_0 - \mu(T-t_0)}{\sigma\sqrt{T-t_0}}\right) = 1 - \Phi\left(\frac{\log k - \log S_0 - \mu(T-t_0)}{\sigma\sqrt{T-t_0}}\right)$$

which is the inthemoney probability in the statistically accurate premium for *S*, the probability associated with the discounted value of the exercise price *k*. The probability of the path of the demeaned asset *Y* will be above the exercise at expiry, at the timing of condition the two variables [where the low path of *Y* emanates from *S*], is

$$\overline{F}_{N(0,1)}\left(u_k'\right) = \overline{F}_{N(0,1)}\left(\frac{\log k - \log S_0 - (r - \frac{1}{2}\sigma^2)(T-t_0)}{\sigma\sqrt{T-t_0}}\right) = 1 - \Phi\left(\frac{\log k - \log S_0 - (r - \frac{1}{2}\sigma^2)(T-t_0)}{\sigma\sqrt{T-t_0}}\right)$$

This latter probability is equal to $N(d_2)$ in the Black-Scholes debased call option formula, where $N(\cdot) = 1 - \Phi(\cdot)$ and $d_2 = u_k' = \left(\log k - \log S_0 - (r - \frac{1}{2}\sigma^2)(T-t_0)\right)/\sigma\sqrt{T-t_0}$ is the normal pivot debased with slack, and this probability is clearly inaccurate [statistical biased] from the slack variable and resulting sink for any time prior to maturity since it is accurate for the wrong asset. The longer the term-to-maturity and the greater the divergence between the growth in the underlying market asset and a bank account, the more inaccurate this probability and the resulting debased premium. The differences between the path of *S* and the paths of *Y* are a direct result of compounding slack for which demeans the variable *S* to *Y* as an exponential sink. The basis function, and hence the crinkliness of the paths, as stochastic variation and sign changes above and below the mean drift, are exactly the same; no return and full market risk.



That of the market asset *S* is the highest simulated path and those of the demeaned asset *Y*, as successively conditioned on the market asset ($S_0 = Y_0$) are those emanating from the path of *S* with smaller gradients [the debased drift].

The debased option premium is the option priced for the low path of the demeaned asset at the time at which the path of the demeaned asset is conditioned on the market asset ($S_0 = Y_0$), i.e., the premium which applies to the time from which any of those paths emanate from below the path of the market asset. The expected value of the divergence between the two paths at option expiry then determines the statistical bias and inaccuracy in the debased price. In the first of the grid of graphics, this is the difference contained within the ellipse. The inthemoney probability in the debased premium relates to the low path while the correct probability of the option relates to the high path. These are the probabilities contained in the premiums associated with the present value of the exercise price; it should be clear, then, that the debased call price underprices the upside and the debased put price overprices the downside [calls at the debased price are statistically biased downwards (statistically underpriced) and puts upwards (statistically overpriced)]. This should make it blatantly clear, if it was not already, that the option premium priced for *Y* [the debased price] is wrong for an option written for *S*. It should also make it clear that the volatility skew resulting from backing out debased prices equated to market prices is a direct result of the compounded slack in the debased price [as a function of term-to-maturity]. The skew is approximately hyperbolic in the logarithm of the exercise price.

The backwards deconvolution recurrence for the delta of the debased option $h^{debased}$

$$h^{debased}(k,T;S_0,t_0) = e^{-rdt}\int_0^{\infty} y \times f_{X'}(y) h^{debased}(k,T;S_0 \times y, t_0 + dt)dy$$

This is exactly that for the option delta of the statistically accurate option premium but with the basis function *X* debased with slack Δ to X' and is the continuous time equivalent to either $h_0^{debased} = vp'uh_u^{debased} + vq'dh_d^{debased}$ debased with probability primed with slack or $h_0^{debased} = vpu'h_{u'}^{debased} + vqd'h_{d'}^{debased}$ debased with step ratios primed with slack [both original to this thesis]. Differentiating the deconvolution recursion of the debased premium for the recursion in the option delta of the debased premium

$$\begin{aligned}
h^{debased}(k,T;Y_0,t_0) &= \frac{\partial}{\partial Y_0} V^{debased}(k,T;Y_0,t_0) \\
&= \frac{\partial}{\partial Y_0} e^{-rdt}\int_0^{\infty} f_{X'}(y) V^{debased}(k,T;Y_0 \times y, t_0 + dt)dy \\
&= e^{-rdt}\int_0^{\infty} f_{X'}(y) \frac{\partial}{\partial Y_0} V^{debased}(k,T;Y_0 \times y, t_0 + dt)dy \\
&= e^{-rdt}\int_0^{\infty} f_{X'}(y) \frac{\partial Y_0 y}{\partial Y_0} \frac{\partial}{\partial Y_0 y} V^{debased}(k,T;Y_0 \times y, t_0 + dt)dy \\
&= e^{-rdt}\int_0^{\infty} y \times f_{X'}(y) h^{debased}(k,T;Y_0 \times y, t_0 + dt)dy
\end{aligned}$$

i.e.,

$$h^{debased}(k,T;Y_0,t_0) = e^{-rdt}\int_0^{\infty} y \times f_{X'}(y) h^{debased}(k,T;Y_0 \times y, t_0 + dt)dy$$

where $X' \sim \log N(\mu' dt, \sigma^2 dt) \sim \log N\left((r - \frac{1}{2}\sigma^2)dt, \sigma^2 dt\right)$ so

$$f_{X'}(y) = \frac{1}{y\sigma\sqrt{2\pi\, dt}} e^{-\frac{1}{2}\frac{[\log y - (r-\frac{1}{2}\sigma^2)dt]^2}{\sigma^2 dt}}$$

and

$$h^{debased}(k,T;Y_0,t_0) = e^{-rdt}\int_0^{\infty} \frac{1}{\sigma\sqrt{2\pi\, dt}} e^{-\frac{1}{2}\frac{[\log y-(r-\frac{1}{2}\sigma^2)dt]^2}{\sigma^2 dt}} h^{debased}(k,T;Y_0\times y,t_0+dt)dy$$

The recursion can similarly be unfolded to the forward variables at option maturity for payoff in the demeaned asset *Y* [the wrong asset] for which the premium and deltas are calculated [and consequently for the wrong probabilities] subject to the initial condition that the demeaned asset is in parity with the spot price of the market asset, i.e., conditional on $Y(t_0) = S_0$.

Solving the recurrence for the DELTA of the debased option $h^{debased}$

Similarly to the option premium, the backwards recursion is solved by unfolding it to the forward variables at option maturity $T = t_0 + ndt$. At each step of dt , this deconvolves a basis component from the delta and convolves it into the density of the forward distribution. For the first infinitesimal from time t_0

$$h^{debased}(k,T;S_0,t_0) = e^{-rdt}\int_0^{\infty} y\times f_{X'}(y)h^{debased}(k,T;S_0\times y,t_0+dt)dy \quad \text{where } X_j' \sim \log N\left((r-\tfrac{1}{2}\sigma^2)dt,\sigma^2 dt\right)$$

unfolding to option expiry at steps of *dt*

$$\begin{aligned}
&h^{debased}(k,T;S_0,t_0)\\
&= e^{-rdt}\int_0^{\infty} y_1\times f_{X'}(y_1)h^{debased}(k,T;S_0\times y_1,t_0+dt)dy_1\\
&= e^{-rdt}\int_0^{\infty} y_2\times f_{X'}(y_2)e^{-rdt}\int_0^{\infty} y_1\times f_{X'}(y_1)h^{debased}(k,T;S_0\times y_1\times y_2,t_0+2dt)dy_1dy_2\\
&= e^{-rdt}\int_0^{\infty} y_3\times f_{X'}(y_3)e^{-rdt}\int_0^{\infty} y_2\times f_{X'}(y_2)e^{-rdt}\int_0^{\infty} y_1\times f_{X'}(y_1)h^{debased}(k,T;S_0\times y_1\times y_2\times y_3,t_0+3dt)dy_1dy_2dy_3\\
&...\\
&= e^{-rdt}\int_0^{\infty} y_n\times f_{X'}(y_n)...e^{-rdt}\int_0^{\infty} y_2\times f_{X'}(y_2)e^{-rdt}\int_0^{\infty} y_1\times f_{X'}(y_1)h^{debased}(k,T;S_0\times y_1\times y_2\times...\times y_n,t_0+ndt)dy_1dy_2...dy_n
\end{aligned}$$

Hence at maturity, the unfolded recursion for the option delta is

$$\begin{aligned}
&h^{debased}(k,T;S_0,t_0)\\
&\quad = e^{-nrdt}\int_0^{\infty} f_{X'}(y_n)...\int_0^{\infty} f_{X'}(y_2)\int_0^{\infty} f_{X'}(y_1)\times y_1\times y_2\times...\times y_n\times h^{debased}(k,T;S_0\times y_1\times y_2\times...\times y_n,t_0+ndt)dy_1dy_2...dy_n
\end{aligned}$$

which, with $n = (T-t_0)/dt$ so that $T = t_0 + ndt$, can be written as

$$\begin{aligned}
&h^{debased}(k,T;S_0,t_0)\\
&\quad = e^{-n(T-t_0)}\int_0^{\infty} f_{X'}(y_n)...\int_0^{\infty} f_{X'}(y_2)\int_0^{\infty} f_{X'}(y_1)\times y_1\times y_2\times...\times y_n\times h^{debased}(k,T;S_0\times y_1\times y_2\times...\times y_n,T)dy_1dy_2...dy_n
\end{aligned}$$

and $x = S_0\times y_1\times y_2\times...\times y_n$ for the statistically accurate premium with the X' debased with the slack variable

$$\begin{aligned}
Y(T) &= S_0\prod_0^n X_j'dt\\
&\sim \log N\left(\log S_0 + (r-\tfrac{1}{2}\sigma^2)ndt,\sigma^2 ndt\right)\\
&\sim \log N\left(\log S_0 + (r-\tfrac{1}{2}\sigma^2)(T-t_0),\sigma^2(T-t_0)\right)
\end{aligned}$$

which is the forward transition distribution of the debased risky asset *Y*, conditional on the spot price of the market asset, which has density function $f_{Y(T)|Y(t)}(x\,|\,S_0)$. The unfolded recursion for the option delta is

$$h^{debased}(k,T;S_0,t_0) = e^{-r(T-t_0)}\int_0^{\infty} \frac{x}{S_0} f_{Y(T)|Y(t_0)}(x\,|\,S_0)\times h^{debased}(k,T;x,T)dx$$

subject to the same call boundary condition as the statistically accurate call option delta, one for a long transaction if inthemoney and zero otherwise, but with respect to the demeaned asset *Y* being or not being inthemoney at option expiry

$$h^{debased}(k,T;Y_T,T)=\begin{cases}1 & if\ \ Y_T>k\\ 0 & if\ \ Y_T<k\end{cases}$$

i.e., the asset *Y* being in or out of the money (1 if it is 0 if it isn't) calculated for a forward transition of length $T-t_0$ from the spot price S_0 at time t_0 with no market return within the forward transition distribution. This corresponds, almost one-for-one with the binomial results debased either by varying the step probabilities or the step ratios, although the latter more closely corresponds to continuous time. The debased call option delta is then

$$\begin{aligned}h^{debased}(k,T;S_0,t_0)&=e^{-r(T-t_0)}\int_0^k \frac{x}{S_0}f_{Y(T)|Y(t_0)}(x\,|\,S_0)\times 0\times dx+e^{-r(T-t_0)}\int_k^\infty \frac{x}{S_0}f_{Y(T)|Y(t_0)}(x\,|\,S_0)\times 1\times dx\\&=e^{-r(T-t_0)}\int_k^\infty \frac{x}{S_0}f_{Y(T)|Y(t_0)}(x\,|\,S_0)dx\end{aligned}$$

which evaluates to that as previously

$$h^{debased}(k,T;S_0,t_0)=\overline{F}_U\left(u_k'-\sigma\sqrt{T-t_0}\right)$$

Subject to the put boundary condition, the delta is negative one for a short transaction if on the region for which the asset is inthemoney and zero otherwise, with respect to the sample space of the option expiry for the demeaned asset *Y* rather than the market asset *S*

$$h^{debased}(k,T;Y_T,T)=\begin{cases}0 & if\ \ Y_T>k\\ -1 & if\ \ Y_T<k\end{cases}$$

the debased put option delta is then

$$\begin{aligned}h^{debased}(k,T;S_0,t_0)&=e^{-r(T-t_0)}\int_0^k \frac{x}{S_0}f_{Y(T)|Y(t_0)}(x)\times(-1)\times dx+e^{-r(T-t_0)}\int_k^\infty \frac{x}{S_0}f_{Y(T)|Y(t_0)}(x)\times 0\times dx\\&=-e^{-r(T-t_0)}\int_0^k \frac{x}{S_0}f_{Y(T)|Y(t_0)}(x)dx\end{aligned}$$

which evaluates to

$$h^{debased}(k,T;S_0,t_0)=-F_U\left(u_k'-\sigma\sqrt{T-t_0}\right)$$

as per differentiating the debased put option premium partially with respect to the market spot price. **These deltas carry no return and have no future value with respect to the market.**

The Backward Equation of the Debased Option Premium

The debased option premium which is priced for the demeaned asset *Y* rather than the market asset to deliver *S* has backward equation

$$rV_0-\frac{\partial V_0}{\partial t_0}=\tfrac{1}{2}\sigma^2Y_0^2\frac{\partial^2V_0}{\partial Y_0^2}+rY_0\frac{\partial V_0}{\partial Y_0}$$

which is derived in exactly the same way as for the statistically accurate premium but simply with respect to the wrong [redundant oxymoron] asset, the demeaned asset *Y*, rather than the market asset *S* upon which the value of the derivative depends. So similarly to the PDE for the statistically accurate premium, there are numerous methods of deriving the debased version of the PDE of the option premium, which is simply the backward equation of the demeaned and redundant asset *Y*, rather than the market asset *S*, with a source for time decay in cash time value.

The PDE of the debased premium is simply that of the accurate premium that results by debasing the drift with the slack variable and consequently simply that of the accurate premium with an additional term in delta and the slack variable ($\Delta Y_0 h_0$) debased from the flux which has the effect of biasing the premium (making it unaccurate for *S*) from which the resulting statistical mispricing then kills any return that a replicating asset may earn and makes all statistical functions related to the option expiry for *S* statistically biased and inaccurate.

So the derivations here are exactly the same except for the place in the mathematics where the slack variable Δ features, implicitly or explicitly. For example, **(1)** the backwards deconvolution recursion for the density, which

is transformed to that of the premium, is with respect to the debased basis density functions X' rather that X and hence where the slack variable features, from which V is expanded to the debased PDE. On the other hand, **(2)** the backwards PDE of the demeaned asset Y, it is the exponential drift of the convection term that is debased to $r-\frac{1}{2}\sigma^2$ with Δ, and then transformed from f to V with no change in the backward equation except for the term for decay in the discount factor [which adds the source in r] similarly to that for S. Further, **(3)** it is also obtained from backing it out of the portfolio return components, which is how the debased equation is [exclusively] derived by return-neutral valuation whereby the slack variable Δ is introduced via the constraint placed on the portfolio flux and then the option PDE is backed out from the portfolio components after the returns have been debased with this constraint that kills the return by fixing it flat at cash [for flat portfolio convexity], i.e., the returns of the portfolio are debased with the constraint $d\Pi_t = r\Pi_t dt$ [which debases the term $\Delta Y_0 h_0$] and then the resulting debased PDE is backed out which relates to the debased option premium [which is accurate for the demeaned asset Y] not the premium accurate for the market asset to deliver S. Each of these are considered now.

Proof 1: deconvolution integral recursion in the premium with basis components primed with slack

From the backwards convolution recurrence of the option premium with basis components X' debased with slack

$$V^{debased}(k,T;S_0,t_0) = e^{-rdt}\int_0^\infty f_{X'}(y)V^{debased}(k,T;S_0\times y,t_0+dt)dy$$

similarly as the recursion for the accurate premium, expanding

$$V^{debased}(k,T;S_0\times y,t_0+dt) = V^{debased}(k,T;e^{\log S_0+\log y},t_0+dt)$$

in a bivariate power series around t_0 and $\log S_0$

$$V^{debased}(k,T;S_0,t_0) = e^{-rdt}\int_0^\infty f_{X'}(y)\left\{1+(dt)\frac{\partial}{\partial t_0}+(\log y)\frac{\partial}{\partial \log S_0}+\tfrac{1}{2}(\log y)^2\frac{\partial^2}{(\partial \log S_0)^2}\right\}V^{debased}(k,T;S_0,t_0)dy+o(dt)$$

and taking the debased basis integral inside the expansion

$$V^{debased}(k,T;S_0,t_0)$$
$$= e^{-rdt}\left\{1+(dt)\frac{\partial}{\partial t_0}+\left(\int_0^\infty f_{X'}(y)(\log y)dy\right)\frac{\partial}{\partial \log S_0}+\tfrac{1}{2}\left(\int_0^\infty f_{X'}(y)(\log y)^2dy\right)\frac{\partial^2}{(\partial \log S_0)^2}\right\}V^{debased}(k,T;S_0,t_0)+o(dt)$$

This is where the derivation differs from that of the accurate premium. Since the basis density functions are primed with slack,

$$\begin{aligned}\log X' &\sim N\left(\mu' dt,\sigma^2 dt\right)\\ &\sim N\left((\mu-\Delta)dt,\sigma^2 dt\right)\\ &\sim N\left((r-\tfrac{1}{2}\sigma^2)dt,\sigma^2 dt\right)\end{aligned}$$

so the statistical expectations for the return is null at cash while the second moment of return is full market risk net of $o(dt)$ terms

(1) $\int_0^\infty f_{X'}(y)(\log y)dy = \mu' dt = (\mu-\Delta)dt = (r-\tfrac{1}{2}\sigma^2)dt$

(2) $\int_0^\infty f_{X'}(y)(\log y)^2 dy = \sigma^2 dt + \left((r-\tfrac{1}{2}\sigma^2)dt\right)^2 = \sigma^2 dt + o(dt)$

Hence

$$V^{debased}(k,T;S_0,t_0) = e^{-rdt}\left\{1+(dt)\frac{\partial}{\partial t_0}+\left((r-\tfrac{1}{2}\sigma^2)dt\right)\frac{\partial}{\partial \log S_0}+\tfrac{1}{2}(\sigma^2 dt)\frac{\partial^2}{(\partial \log S_0)^2}\right\}V^{debased}(k,T;S_0,t_0)+o(dt)$$

and expanding the discount factor e^{-rdt}

$$V^{debased}(k,T;S_0,t_0)=\left\{1+(dt)\frac{\partial}{\partial t_0}+\left((r-\tfrac{1}{2}\sigma^2)dt\right)\frac{\partial}{\partial \log S_0}+\tfrac{1}{2}(\sigma^2 dt)\frac{\partial^2}{(\partial \log S_0)^2}\right\}V^{debased}(k,T;S_0,t_0)$$
$$-(rdt)V^{debased}(k,T;S_0,t_0)+o(dt)$$

Cancelling $V^{debased}(k,T;S_0,t_0)$ from both sides and dividing through by *dt*

$$rV^{debased}(k,T;S_0,t_0)-\frac{\partial}{\partial t_0}V^{debased}(k,T;S_0,t_0)$$
$$=\tfrac{1}{2}\sigma^2\frac{\partial^2}{(\partial \log S_0)^2}V^{debased}(k,T;S_0,t_0)+(r-\tfrac{1}{2}\sigma^2)\frac{\partial}{\partial \log S_0}V^{debased}(k,T;S_0,t_0)+\frac{o(dt)}{dt}$$

and with

(1) $\frac{\partial}{\partial \log S_0}=S_0\frac{\partial}{\partial S_0}$ and **(2)** $\frac{\partial^2}{(\partial \log S_0)^2}=S_0{}^2\frac{\partial^2}{\partial S_0{}^2}+S_0\frac{\partial}{\partial S_0}$

the PDE of the debased premium becomes

$$rV^{debased}(k,T;S_0,t_0)-\frac{\partial}{\partial t_0}V^{debased}(k,T;S_0,t_0)=\tfrac{1}{2}\sigma^2 S_0{}^2\frac{\partial^2}{\partial S_0{}^2}V^{debased}(k,T;S_0,t_0)+rS_0\frac{\partial}{\partial S_0}V^{debased}(k,T;S_0,t_0)+\frac{o(dt)}{dt}$$

Letting $dt\to 0$ so that $o(dt)/dt\to 0$

$$rV^{debased}(k,T;S_0,t_0)-\frac{\partial}{\partial t_0}V^{debased}(k,T;S_0,t_0)=\tfrac{1}{2}\sigma^2 S_0{}^2\frac{\partial^2}{\partial S_0{}^2}V^{debased}(k,T;S_0,t_0)+rS_0\frac{\partial}{\partial S_0}V^{debased}(k,T;S_0,t_0)$$

which is the PDE of the debased option premium

$$rV_t'-\frac{\partial V_t'}{\partial t}=\tfrac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V_t'}{\partial S_t^2}+rS_t\frac{\partial V_t'}{\partial S_t}$$

Note that this is simply that of the accurate premium with slack in the drift, i.e., with debased drift μ', $\mu+\frac{1}{2}\sigma^2-\Delta=r$

$$rV_t'-\frac{\partial V_t'}{\partial t}=\tfrac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V_t'}{\partial S_t^2}+(\mu+\tfrac{1}{2}\sigma^2-\Delta)S_t\frac{\partial V_t'}{\partial S_t}$$

that debases the flux, and hence how the constraint $d\Pi_t=r\Pi dt$ surreptitiously introduces the slack variable. So with this term separated out, the equation is that for the accurate premium with the additional term $\Delta S_t h_t$ which is surreptitiously introduced into the PDE by holding the portfolio returns fixed flat at *r*

$$rV_t'-\frac{\partial V_t'}{\partial t}=\tfrac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V_t'}{\partial S_t^2}+(\mu+\tfrac{1}{2}\sigma^2)S_t\frac{\partial V_t'}{\partial S_t}-\Delta S_t\frac{\partial V_t'}{\partial S_t} \qquad f_0'=vpf_u'+vqf_d'-\Delta(u-d)S_0h_0'$$

where $\Delta=\mu+\frac{1}{2}\sigma^2-r$. It is this term $\Delta S_t h_t$ that inadvertently debases the option portfolio and premium, as a result of [artificially] holding the portfolio flux fixed flat at *r* with the constraint $d\Pi_t=r\Pi dt$, and that which the replicating asset then tries to subsidize and earn back the resulting mispricing in the premium to keep the portfolio debased from the benefits of trading the market asset and instead held flat at cash and free from convex flux in market return. The equivalent term in the binomial context when *p* is varied with a slack variable Δ is $\Delta(u-d)S_0h_0'$, i.e., the recursion for the debased premium $f_0'=vp'f_u'+vq'f_d'$ with the slack separated out of *p* and *q* is $f_0'=vpf_u'+vqf_d'-\Delta(u-d)S_0h_0'$ where the premium f' and delta h' remain primed with the slack variable $\Delta=(pu+qd-1-r)/(u-d)$ so that the debased term is $(pu+qd-1-r)S_0h_0'$. These are mainly analogous, apart from $(\mu+\frac{1}{2}\sigma^2-r)S_th_t'$ being a continuous variable while $(pu+qd-1-r)S_0h_0'$ being discrete.

Proof 2: transforming the backward equation of the demeaned asset *Y*

The backward equation of the demeaned asset *Y* is the same as that for the market asset *S*, with the same total and infinitesimal risk, but with null exponential drift and return. From the Raison D'etre, repeating the mathematics for the demeaned asset and oxymoron risky bank account;

From the backward equation of the demeaned asset *Y* which earns nothing but has full market risk

$$-\frac{\partial}{\partial t} f_{Y(T)|Y(t)}(x \mid Y_t) = \tfrac{1}{2}\sigma^2 Y_t^2 \frac{\partial^2}{\partial Y_t^2} f_{Y(T)|Y(t)}(x \mid Y_t) + rY_t \frac{\partial}{\partial Y_t} f_{Y(T)|Y(t)}(x \mid Y_t)$$

Multiplying by the intrinsic value, which is a function of the forward diffusion variable *x*, and integrating over the range of the sample space of the forward diffusion variable for which the option is inthemoney for the asset for which the option is priced, in this case, the demeaned asset *Y*; for a call option priced for *Y*

$$-\int_k^\infty (x-k)\frac{\partial}{\partial t} f_{Y(T)|Y(t)}(x \mid Y_t)dx = \int_k^\infty (x-k)\left(\tfrac{1}{2}\sigma^2 Y_t^2 \frac{\partial^2}{\partial Y_t^2} f_{Y(T)|Y(t)}(x \mid Y_t) + rY_t \frac{\partial}{\partial Y_t} f_{Y(T)|Y(t)}(x \mid Y_t)\right)dx$$

Taking the integrals inside the partial derivatives which are derivatives in the backward variables

$$-\frac{\partial}{\partial t}\int_k^\infty (x-k)\, f_{Y(T)|Y(t)}(x \mid Y_t)dx = \left(\tfrac{1}{2}\sigma^2 Y_t^2 \frac{\partial^2}{\partial Y_t^2} + rY_t \frac{\partial}{\partial Y_t}\right)\int_k^\infty (x-k)\, f_{Y(T)|Y(t)}(x \mid Y_t)dx$$

and discounting both sides over the forward transition of $T-t$ periods from *T* to *t* at the cash rate *r* with the discount factor $e^{-r(T-t)}$

$$-e^{-r(T-t)}\frac{\partial}{\partial t}\int_k^\infty (x-k)\, f_{Y(T)|Y(t)}(x \mid Y_t)dx = e^{-r(T-t)}\left(\tfrac{1}{2}\sigma^2 Y_t^2 \frac{\partial^2}{\partial Y_t^2} + rY_t \frac{\partial}{\partial Y_t}\right)\int_k^\infty (x-k)\, f_{Y(T)|Y(t)}(x \mid Y_t)dx$$

Taking the discount factor $e^{-r(T-t)}$ inside the partial derivatives; that in the backward time variable adds a source as a diminishing sink

$$\left(r-\frac{\partial}{\partial t}\right)e^{-r(T-t)}\int_k^\infty (x-k)\, f_{Y(T)|Y(t)}(x \mid Y_t)dx = \left(\tfrac{1}{2}\sigma^2 Y_t^2 \frac{\partial^2}{\partial Y_t^2} + rY_t \frac{\partial}{\partial Y_t}\right)e^{-r(T-t)}\int_k^\infty (x-k)\, f_{Y(T)|Y(t)}(x \mid Y_t)dx$$

With $V^{debased}(k,T;Y_t,t) = e^{-r(T-t)}\int_k^\infty (k-x)\, f_{Y(T)|Y(t)}(x \mid Y_t)dx$, this is the PDE of the debased call option premium which is statistically unbiased for the demeaned asset *Y*, the market asset that is debased of market return

$$\left(r-\frac{\partial}{\partial t}\right)V^{debased}(k,T;Y_t,t) = \left(\tfrac{1}{2}\sigma^2 Y_t^2 \frac{\partial^2}{\partial Y_t^2} + rY_t \frac{\partial}{\partial Y_t}\right)V^{debased}(k,T;Y_t,t)$$

i.e.,

$$rV^{debased}(k,T;Y_t,t) - \frac{\partial}{\partial t}V^{debased}(k,T;Y_t,t) = \tfrac{1}{2}\sigma^2 Y_t^2 \frac{\partial^2}{\partial Y_t^2}V^{debased}(k,T;Y_t,t) + rY_t \frac{\partial}{\partial Y_t}V^{debased}(k,T;Y_t,t)$$

subject to the initial condition that the demeaned asset is in parity with the spot price of the market asset ($Y(t_0) = S_0$) and the boundary condition of the value of the call option to buy the demeaned asset *Y* if it is greater than *k* at the maturity date

$$V^{debased}\big(k,T;Y(T),T\big) = \max\big(Y(T)-k,0\big)$$

for which the probability that *Y* will expire inthemoney is a junk and biased probability [as the premium is] for the probability that *S* will expire inthemoney, as per the junk probabilities of $N(d_1)$ in Black and Scholes. This is the PDE of the option derivative that is accurate for, and only for, the oxymoron risky bank account *Y*. It is statistically biased and inaccurate for all other assets and therefore is a PDE of a price of an option derivative that systematically misprices an option derivative on the market asset *S*, without any qualifications whatsoever. The derivation of the PDE of the debased premium for a put option is similar (omitted) and the PDE is identical apart from the boundary condition

$$V^{debased}\big(k,T;Y(T),T\big) = \max\big(k-Y(T),0\big)$$

Note that the demeaned asset *Y* is the market asset *S* with a sink in the slack variable $Y(T) = S(T)\mathrm{e}^{-\Delta T}$ so when this PDE is solved for the boundary condition for the market asset $S(T) = Y(T)\mathrm{e}^{\Delta T}$ to buy, instead of the demeaned asset *Y*, it is a statistical transformation of the PDE which remeans the premium to the undebased price by putting a source on the forward variable and then solves for the statistically accurate price rather than the debased price. The debased price of Black and Scholes is the wrong price. There are four methods of remeaning the demeaned asset and premium [and consequently the PDE] to get the formula for the accurate premium and hence a statistically accurate price from the debased price, i.e., four methods of removing the slack variable from the debased premium. This, transforming the forward variable, is one of them. Remeaning the spot value is another, which is transforming the backward variable and that by reference to market put-call parity. These two methods are equivalent due to the time homogeneity of first order diffusion. The others are putting a source on the exercise price and skewing volatility [i.e., inaccurately and inefficiently with implied volatility]. These are explored later in this book. Although a bit stupid, you could also skew the cash rate in the discount factor applied to the premium. Note that the PDE of the debased premium here, remeaned by skewing volatility with slack to implied volatility, is not a correct PDE because implied volatility is a non-constant function of term-to-maturity $T - t_0$, and the PDE for an option premium on Brownian motion [for *S* or *Y*] applies to constant volatility only, and is hence wrong when parameterized with implied volatility, as people do with the debased formula in practice.

Proof 3: Backing out the backwards recurrence for the debased premium from a debased delta neutral option replicating portfolio

The debased replicating portfolio composed of an offsetting delta position in the underlying market asset and a short position in the option priced at the debased premium is

$$\Pi_0' = S_0 h_0' - V_0' \qquad \text{where} \qquad h_0' = \frac{\partial V_0'}{\partial S_0}$$

where the portfolio, premium, and deltas are primed to signify that they have been debased. Differentiating the portfolio

$$\frac{d\Pi_0'}{dt_0} = (S_0 + dS_0)\frac{dh_0'}{dt_0} + h_0'\frac{dS_0}{dt_0} - \frac{dV_0'}{dt_0}$$

ignoring the delta update this is

$$\frac{d\Pi_0'}{dt_0} = h_0'\frac{dS_0}{dt_0} - \frac{dV_0'}{dt_0}$$

(1) The debased premium is characterized by the constraint that the replicating portfolio changes/dynamics, net of the continuous stream of instantaneous delta rebalances, are held fixed flat at cash by saying [wrongly] that *"delta neutral portfolios must earn the cash rate"*

$$\frac{d\Pi_0'}{dt_0} = rS_0 h_0' - rV_0'$$

i.e., that $d\Pi_0^{debased} = r\Pi_0^{debased} dt_0$ ($d\Pi_0' = r\Pi_0' dt_0$) which is the cash rate on both the market asset and the option. Black and Scholes (1973) guessed and reasoned wrongly, and inadvertently debased derivatives.

(2) To get another expression for the portfolio change, expanding the debased premium $V_0' = V_0'(S_0, t_0)$ in a power series

$$\begin{aligned}\frac{dV_0'}{dt_0} &= \frac{\partial V_0'}{\partial t_0}\frac{dt_0}{dt_0} + \frac{\partial V_0'}{\partial S_0}\frac{dS_0}{dt_0} + \tfrac{1}{2}\frac{\partial^2 V_0'}{\partial S_0^2}\frac{(dS_0)^2}{dt_0} \\ &= \frac{\partial V_0'}{\partial t_0} + \frac{\partial V_0'}{\partial S_0}\frac{dS_0}{dt_0} + \tfrac{1}{2}\sigma^2 S_0^2\frac{\partial^2 V_0'}{\partial S_0^2}\end{aligned}$$

and substituting this into the portfolio derivative and cancelling the debased option delta with the replicating delta

$$\frac{d\Pi_0{}'}{dt_0} = h_0{}' \frac{dS_0}{dt_0} - \left(\frac{\partial V_0{}'}{\partial t_0} + \frac{\partial V_0{}'}{\partial S_0} \frac{dS_0}{dt_0} + \tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 V_0{}'}{\partial S_0^2} \right) = -\frac{\partial V_0{}'}{\partial t_0} - \tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 V_0{}'}{\partial S_0^2}$$

(1) and **(2)** now provides two expressions for an identity in the portfolio differential $\frac{d\Pi_0{}'}{dt_0} \equiv \frac{d\Pi_0{}'}{dt_0}$

$$\frac{d\Pi_0{}'}{dt_0} = rS_0 \frac{\partial V_0{}'}{\partial S_0} - rV_0' \qquad \text{and} \qquad \frac{d\Pi_0{}'}{dt_0} = -\frac{\partial V_0{}'}{\partial t_0} - \tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 V_0{}'}{\partial S_0^2}$$

Equating the two identities for the debased portfolio returns then backs out the backward equation of the debased premium

$$rS_0 \frac{\partial V_0{}'}{\partial S_0} - rV_0' = -\frac{\partial V_0{}'}{\partial t_0} - \tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 V_0{}'}{\partial S_0^2}$$

i.e.,

$$rV^{debased}(k,T;S_0,t_0) - \frac{\partial}{\partial t_0} V^{debased}(k,T;S_0,t_0) = \tfrac{1}{2}\sigma^2 S_0{}^2 \frac{\partial^2}{\partial S_0{}^2} V^{debased}(k,T;S_0,t_0) + rS_0 \frac{\partial}{\partial S_0} V^{debased}(k,T;S_0,t_0)$$

which is null of the benefits of trading the exponential market asset *S* and is an equation that misprices the derivative by the difference between the premium and dependent variable of this equation $V^{debased}$ and that of the premium and dependent variable *V* of the equation with the correct rate of market return, i.e., the correct rate of exponential drift. The PDE and option premium have been backed out of the wrong return components (those flat at cash) for which the consequence is the wrong exponential rate of drift, $r - \frac{1}{2}\sigma^2$, rather than μ. With the constraint $d\Pi_0^{debased} = r\Pi_0^{debased} dt_0$, the PDE has been debased of the term $\Delta S_0 h_0 = (\mu + \frac{1}{2}\sigma^2 - r) S_0 h_0$ which surreptitiously kills the portfolio market flux by debasing the exponential drift. This is then extrapolated as a justification to kill the parameter μ in all derived functions of the replicating portfolio and cashflows, and has been the justification to debase derivatives and discredit intelligent people (who shouldn't have been discredited) and fail PhD theses (which shouldn't have been failed) and regress probability theory (with abysmal nonsense).

Note that the returns for a delta-neutralized portfolio can be set to anything; the premium is then adjusted with slack accordingly so that the premium adjusts to fit whatever the hypothetical returns are. This is because it is not a correct mathematical method of deriving the PDE. For the returns on the items of the replicating portfolio, **(1)** the coefficient of the market asset then becomes the drift of the asset for which the premium is priced and **(2)** the coefficient of the option becomes the rate at which the premium is discounted for present values. The resulting premium is then forced to sink or source the divergence between these coefficients and **(1)** the earnings on the market asset and **(2)** the cash rate for discounting. Only when it is convex flux in market return with market return earned on the market asset and the cash rate on the short option is it correct for the pricing of derivatives, since only then does **(1)** the drift correspond to the market asset for which the derivative depends and does **(2)** the rate at which the premium is discounted correspond to the cash rate for present values. Then, and only then, is the PDE for the premium backed out of returns the correct PDE for the accurate premium, and that sensible for the theory of derivatives.

The only value of risk neutral valuation is for amusement in a book such as *"Riddles in Mathematics"* (1944) by E.P. Northrop, a book written about mathematical and logical problems that appear paradoxical [in a time[6] when nonsense was not tolerated]. It would appear under the heading

"How to prove that $\mu + \frac{1}{2}\sigma^2$ is equal to r when it is not"

or in the context of finance

"how to prove that the markets have no return when they do"

There are many instances where the cash rate does not beat inflation. In these circumstances, it is reasonable to assume that the overall rate for cash is zero. It is instructional to then look at the risk-neutral theory and ask yourself does it still make sense? Does it still make sense to say that the portfolio earns nothing therefore must be risk free (despite being dependent upon σ)? Does it still makes sense to say that the portfolio earns nothing

[6] of war, and cracking the enigma and splitting the atom [and hence victory because the science worked].

therefore must be independent of the risk preferences of investors? Setting the inflation rate to $r-\frac{1}{2}\sigma^2$, and zeroing the exponential drift, the BSM equation becomes

$$-\frac{\partial V_0'}{\partial t_0}=\tfrac{1}{2}\sigma^2 S_0^2\frac{\partial^2 V_0'}{\partial S_0^2}$$

This is not a sensible PDE for which to price an option on a market asset, a premium for 100% of the market risk and 0% of the market return, i.e., the PDE of a variance-only premium. The risk-preferences of any sensible investor would always shun the oxymoron risky bank account, so why should their reasoning be any different for derivatives? The fact that the result of holding the portfolio return fixed flat is the surreptitious pricing of the option for the demeaned asset, rather than the asset itself, results in nothing other than the systematic mispricing of options and the debasement of derivatives.

The Total Derivative of the Option Priced for the Demeaned Asset Y with Time Decay in the Demeaned Asset Y

Consideration is now given to the total derivative for the circumstance whereby each item in the replicating portfolio has been debased with the slack variable Δ, i.e., the replicating asset *S*, the option premium *V*, and the option delta *h*. These are debased to the demeaned asset *Y*, the debased option premium $V^{debased}$ accurate for *Y*, and option delta for this debased premium $h^{debased}$. This is as a result of the constraint $d\Pi_t=r\Pi_t dt$. which then primes Π with slack and all of the items within Π. This is the case whereby the debased option is replicated and offset by the demeaned asset *Y* for which the premium and delta are accurate. This, a debased portfolio with a debased premium trading a demeaned asset, is what return-neutral valuation refers to as "*an alternative Q world for derivative pricing that does not exist [but is true in every world]".* What is really meant by this is *'a debased portfolio that notionally trades the oxymoron risky bank account Y [that earns nothing] to replicate the debased option premium priced for, and only for, this redundant asset [resulting in no net cashflow since everything is void of market return]'*. What is actually meant by *"but is true in every world"* is that *'the portfolio earns nothing no matter what the replicating asset earns'*, i.e., the portfolio remains debased [primed] despite the replicating asset being remeaned.

Debasing the market asset, the option premium, and the option delta with the slack variable $\Delta=\mu+\frac{1}{2}\sigma^2-r$ so that the resulting demeaned asset *Y*, and the asset for which the debased option premium is then priced, has null return net of cash

$$\mu'+\tfrac{1}{2}\sigma^2=\mu-\Delta+\tfrac{1}{2}\sigma^2=\mu-(\mu+\tfrac{1}{2}\sigma^2-r)+\tfrac{1}{2}\sigma^2=r$$

This is synonymous with the result of that by definition of the demeaned asset *Y* for which the items of the replicating portfolio are accurate. The total derivative of the option premium with the debased drift becomes

$$\begin{aligned}\frac{dV_t'}{dt}&=rV_t'-(\mu'+\tfrac{1}{2}\sigma^2)Y_t\frac{\partial V_t'}{\partial Y_t}+\frac{\partial V_t'}{\partial Y_t}\frac{dY_t}{dt}\\&=rV_t'-rY_t\frac{\partial V_t'}{\partial Y_t}+\frac{\partial V_t'}{\partial Y_t}\frac{dY_t}{dt}\end{aligned}$$

$$\frac{dV_t'}{dt}=rV_t'-rY_th_t'+h_t'\frac{dY_t}{dt}\qquad\text{where } h_t'=\frac{\partial V_t'}{\partial S_t}\text{ is the }\textbf{debased instantaneous option delta}$$

and since the demeaned asset has equation

$$\frac{dY_t}{dt}=rY_t+\sigma Y_t\frac{dB_t}{dt}\qquad\text{so that }\frac{dY_t}{dt}-rY_t=\sigma Y_t\frac{dB_t}{dt}$$

for a proportional diffusion in *Y*, the total derivative can be written as

$$\frac{dV_t'}{dt}=rV_t'+\sigma h_t'Y_t\frac{dB_t}{dt}$$

Note that the asset risk, after the asset has been demeaned, remains unchanged; this demeaned asset has no return, net of cash, yet full *dB* market risk at full market volatility σ. This is a financial oxymoron and a redundant financial asset in the context of all other financial theory. For this reason the demeaned asset *Y* is also, correctly

and accurately, called an oxymoron risky bank account and a redundant asset [similarly to the option for which it is priced].

Proof: total derivative of option premium with time decay in the demeaned asset *Y*

The demeaned asset (the oxymoron risky bank account) for which the debased premium is priced, and here for which replicates the debased premium in $\Pi_0^{debased} = Y_0 h_0^{debased} - V_0^{debased}$, has transition distribution

$$Y(T) \sim \log N\left(\log Y_0 + (r - \tfrac{1}{2}\sigma^2)(T - t_0),\ \sigma^2(T - t_0)\right)$$

and diffusion equation

$$\frac{dY_0}{dt_0} = rY_0 + \sigma Y_0 \frac{dB_0}{dt_0}$$

The debased premium is that which is priced (statistically) for the demeaned asset conditional on $Y(t_0) = Y_0$ for arbitrary Y_0

$$V_0' = V^{debased}(k, T; Y_0, t_0) = \begin{cases} e^{-r(T-t_0)} \int_k^{\infty} (x - k) f_{Y(T)|Y(t_0)}(x \mid Y_0) dx \\ e^{-r(T-t_0)} \int_0^{k} (k - x) f_{Y(T)|Y(t_0)}(x \mid Y_0) dx \end{cases} = \begin{cases} Y_0 \overline{F}_U\left(u_k' - \sigma\sqrt{T - t_0}\right) - k e^{-r(T-t_0)} \overline{F}_U\left(u_k'\right) \\ k e^{-r(T-t_0)} F_U\left(u_k'\right) - Y_0 F_U\left(u_k' - \sigma\sqrt{T - t_0}\right) \end{cases}$$

which has the backward equation

$$-\frac{\partial V_0'}{\partial t_0} = \tfrac{1}{2}\sigma^2 Y_0^2 \frac{\partial^2 V_0'}{\partial Y_0^2} + rY_0 \frac{\partial V_0'}{\partial Y_0} - rV_0'$$

as conditional on $Y(t_0) = Y_0$. Expanding the total derivative of the option premium $V' = V'(t_0, Y_0)$, which here notionally depends on the demeaned asset rather than the market asset for time decay, in a power series and substituting in the backward equation

$$\begin{aligned} \frac{dV_0'}{dt_0} &= \frac{\partial V_0'}{\partial t_0}\frac{dt_0}{dt_0} + \frac{\partial V_0'}{\partial Y_0}\frac{dY_0}{dt_0} + \tfrac{1}{2}\frac{\partial^2 V_0'}{\partial Y_0^2}\frac{(dY_0)^2}{dt_0} \\ &= \frac{\partial V_0'}{\partial t_0} + \frac{\partial V_0'}{\partial Y_0}\frac{dY_0}{dt_0} + \tfrac{1}{2}\sigma^2 Y_0^2 \frac{\partial^2 V_0'}{\partial Y_0^2} \\ &= -\left(\tfrac{1}{2}\sigma^2 Y_0^2 \frac{\partial^2 V_0'}{\partial Y_0^2} + rY_0 \frac{\partial V_0'}{\partial Y_0} - rV_0'\right) + \frac{\partial V_0'}{\partial Y_0}\frac{dY_0}{dt_0} + \tfrac{1}{2}\sigma^2 Y_0^2 \frac{\partial^2 V_0'}{\partial Y_0^2} && \text{substituting the PDE} \\ &= rV_0' - rY_0 \frac{\partial V_0'}{\partial Y_0} + \frac{\partial V_0'}{\partial Y_0}\frac{dY_0}{dt_0} && \text{cancelling } \tfrac{1}{2}\sigma^2 Y_0^2 \frac{\partial^2 V_0'}{\partial Y_0^2} \end{aligned}$$

and given that $\frac{dY_0}{dt_0} - rY_0 = \sigma Y_0 \frac{dB_0}{dt_0}$ this can be written as

$$\frac{dV_0'}{dt_0} = rV_0' + \sigma Y_t \frac{\partial V_0'}{\partial Y_0}\frac{dB_0}{dt_0}$$

This is the total derivative for the circumstance whereby each item in the replicating portfolio has been debased with the slack variable Δ, and the replicating asset that replicates the debased premium has notionally been left debased for the replication, and hence the premium debased with slack [and hence accurate for, and only for, *Y*] is notionally replicated by trading *Y*. This, a debased portfolio that notionally trades the demeaned asset [for killed portfolio cashflows] is the so called *"alternative world Q for derivative pricing that does not exist"*. Consideration is given to the case whereby the replicating asset is remeaned, whilst keeping the premium and deltas debased [for the so called *"return to the real P world"* whereby the portfolio stills earns nothing despite the asset being remeaned] after the portfolio returns and net replication cashflows are discussed for the case where the replicating asset remains demeaned and debased with the slack variable.

■

Noting that by remeaning the demeaned asset (removing the slack from the mean of the asset) from *Y* to *S* whilst keeping the option premium and deltas debased (misconstrued by return-neutral valuation as *"a return to the real P world that then has no risk"*) results in a total derivative of the debased premium as

$$\frac{dV_0'}{dt_0} = rV_0' + \left(\mu + \tfrac{1}{2}\sigma^2 - r\right)h_0'S_0 + \sigma h_0'S_0\frac{dB_0}{dt_0}$$

where the premium V' and deltas h' are primed with slack while the diffusion equation of the asset is not, so that the premium V' debased with Δ then has time decay in the market asset rather than the demeaned asset for which it is priced. Remeaning the replicating asset [by removing the slack variable] is considered after the replicating portfolio cashflows for which the debased premium [which is priced for *Y*] is replicated by the demeaned asset *Y* [for which the debased premium is accurate] which is considered now [as for the binomial walk].

The Return Components of the Debased Option Replicating Portfolio that Replicates with the Demeaned Asset Y

The portfolio returns $d\Pi_t' = (\mu' + \tfrac{1}{2}\sigma^2)h_t'Y_t dt - rV_t'dt$ with a market return of the market asset debased with slack μ' becomes

$$\begin{aligned} h_t'dY_t - dV_t' &= (\mu' + \tfrac{1}{2}\sigma^2 - \Delta)h_t'Y_t dt - rV_t'dt \\ &= rh_t'Y_t dt - rV_t'dt \\ &= r(h_t'Y_t - rV_t')dt \\ &= r\Pi_t'dt \end{aligned}$$

i.e., $d\Pi_t' = r\Pi_t'dt$ net of the rebalance. With the delta update with the demeaned asset *Y* rather than the market asset *S*, the total derivative of the portfolio becomes

$$\begin{aligned} d\Pi_t' &= r\Pi_t'dt + (Y_t + dY_t)dh_t' \\ &= r\Pi_t'dt + (Y_t + \sigma^2 Y_t^2\Gamma_t')dh_t' \end{aligned}$$

net of o(*dt*) variation.

Proof: debased portfolio returns with time decay in the demeaned asset *Y*

In the context of the results derived for the accurate premium and replicating portfolio by PART II of this book of this thesis, this proof is somewhat trivial since it is simply the results of the replicating portfolio for the accurate premium with everything debased whereby the traded asset is the demeaned asset *Y* for which the debased premium is accurate. Exactly the same results for the replicating portfolio for the accurate premium for *S* apply except that all cashflows are null of market return, i.e., every occurrence of μ is replaced with $\mu' = \mu - \Delta$ [wherever it occurs]. The option priced for *Y* is replicated by trading *Y* with the deltas of the option accurate for *Y*. The return on the offsetting asset is the cash rate and the time decay in the option premium depends on the asset which returns the cash rate, for the premium which is priced accurately for the asset which returns only the cash rate.

The total derivative of the option priced for the demeaned asset *Y* and with time decay in the demeaned asset *Y* is

$$\frac{dV_t'}{dt} = rV_t' - rY_t\frac{\partial V_t'}{\partial Y_t} + \frac{\partial V_t'}{\partial Y_t}\frac{dY_t}{dt}$$

The portfolio whereby all items have been debased with the slack variable is $\Pi_t^{debased} = h_t^{debased}Y_t - V_t^{debased}$; substituting the total derivative of the option premium into the total portfolio derivative

$$\frac{d\Pi_{t,T}'}{dt} = h_{t,T}'\frac{dY_t}{dt} - \left(rV_t' - rY_t\frac{\partial V_t'}{\partial Y_t} + \frac{\partial V_t'}{\partial Y_t}\frac{dY_t}{dt}\right) + (Y_t + dY_t)\frac{dh_{t,T}'}{dt} \qquad \text{where} \quad h_t' = \frac{\partial V_t'}{\partial Y_t}$$

$$\begin{aligned} &= rh_t'Y_t - rV_t' + (Y_t + dY_t)\frac{dh_{t,T}'}{dt} \\ &= r\Pi_t' + (Y_t + dY_t)\frac{dh_{t,T}'}{dt} \end{aligned}$$

after canceling the replicating delta in the demeaned asset with the delta of the option priced for the demeaned asset. Net of the update, this is $d\Pi'_{t,T} = r\Pi'_{t,T}dt$. Alternatively, noting that the total derivative of the option can be written as $dV'_t = rV'_t dt + \sigma Y_t h'_t dB_t$ and diffusion equation of the demeaned asset is $dY_t = rY_t dt + \sigma Y_t dB_t$ so $h'_{t,T}dY_t$ can be written as $h'_t dY_t = rh'_t Y_t dt + \sigma h'_t Y_t dB_t$ so the total change in the replicating portfolio can be expressed as

$$\frac{d\Pi'_{t,T}}{dt} = \left(rh'_t Y_t + \sigma h'_t Y_t \frac{dB_t}{dt} \right) - \left(rV'_t + \sigma Y_t \frac{\partial V'_t}{\partial Y_t}\frac{dB_t}{dt} \right) + (Y_t + dY_t)\frac{dh'_{t,T}}{dt} \qquad \text{where} \quad h'_t = \frac{\partial V'_t}{\partial Y_t}$$

$$= h'_t Y_t - rV'_t + (Y_t + dY_t)\frac{dh'_{t,T}}{dt}$$

$$= r\Pi'_t + (Y_t + dY_t)\frac{dh'_{t,T}}{dt}$$

showing that the position in the demeaned asset (the oxymoron risky bank account, *S* debased with slack Δ) earns nothing net of cash $h_t E\left(dY_t \mid Y_t\right) = r\, h_t Y_t dt$, as does the option $E\left(dV'_t \mid Y_t\right) = rV'_t dt$ which is accurate only for that debased asset, and there is no cross subsidization between the debased premium and the demeaned asset since the debased premium and delta is accurate for the demeaned asset *Y*. This is what return-neutral valuation calls *the alternative Q world that does not exist which has no risk for derivative pricing but is true in every world*. It is, rather, just the case whereby all items of the replicating portfolio have been debased with the slack variable Δ, the asset, the premium and the deltas, which then voids all portfolio cashflow of market return. No option and no market return, and nothing to show for a hellovalot of trading.

The NPV of Cashflows for the Debased Option Replicating Portfolio that Replicates with the Demeaned Asset Y

For the option that is priced for the demeaned asset *Y* and which is replicated by trading the demeaned asset for which it is accurate, the differential return on the portfolio is debased and fixed flat at the cash rate *r* because all items within the portfolio are debased and primed with slack; the premium, the premiums delta, and the replicating asset ($S' = Y$). This means killed net cashflows and is the *"alternative Q world that does not exist for derivative pricing"* where the replicating asset remains debased with slack. The total change in the debased portfolio $\Pi'_t = h'_t Y_t - V'_t$, inclusive of the delta update and rebalance with the demeaned asset $(Y_t + dY_t)dh'_t$, is then

$$d\Pi'_t = r\Pi'_t dt + (Y_t + dY_t)dh'_t$$

where $V'_t = e^{-r(T-t)}E\left(\max\left(Y(T)-k,0\right) \mid Y(t) = Y_t\right)$ and $V'_T = E\left(\max\left(Y(T)-k,0\right) \mid Y(T) = Y_T\right)$ for a call option and $V'_t = e^{-r(T-t)}E\left(\max\left(k-Y(T),0\right) \mid Y(t) = Y_t\right)$ and $V'_T = E\left(\max\left(k-Y(T),0\right) \mid Y(T) = Y_T\right)$ for a put option. The total derivative of the portfolio can be written as

$$d\Pi'_t = r\Pi'_t dt + (Y_t + \sigma^2 Y_t^2 \Gamma'_t)dh'_t$$

net of further $o(dt)$ variation in the rebalance. The delta update is then a buy or a sell in the demeaned asset *Y*, and **hence** time decay in the debased premium is with respect to the demeaned asset for which it is accurate. Dividing the differential through by *dt* and moving all terms in the portfolio value to the left-hand-side gives a first order linear differential equation for the portfolio value in terms of a stochastic input of the demeaned asset (plus an Ito-like term) times the rate of change in the delta for the instantaneous delta rebalances

$$\frac{d\Pi'_t}{dt} - r\Pi'_t = (Y_t + \sigma^2 Y_t^2 \Gamma'_t)\frac{dh'_t}{dt}$$

The integrating factor is $IF(t) = e^{-rt}$. Upon multiplication by the integrating factor, which puts all terms in present values, the equation can be differentially factorized giving the differential equation in soluble form for direct integration

$$\frac{d}{dt}\left(e^{-rt}\Pi'_t\right) = e^{-rt}(Y_t + \sigma^2 Y_t^2 \Gamma'_t)\frac{dh'_t}{dt}$$

and integrating over the replication period to option expiration at time *T*

$$\int_0^T \frac{d}{dt}\left(e^{-rt}\Pi_t'\right)dt = \int_0^T e^{-rt}(Y_t + \sigma^2 Y_t^2 \Gamma_t')\frac{dh_t'}{dt}dt$$

i.e.,

$$e^{-rT}\Pi_T' - \int_0^T e^{-rt}(Y_t + \sigma^2 Y_t^2 \Gamma_t')dh_t' - \Pi_0' = 0$$

which is the NPV of the cashflows of the replicating portfolio for that which replicates the debased premium by notionally trading the [redundant] demeaned asset *Y* for which the premium is accurate. In terms of the portfolio contents $\Pi_t' = h_t' Y_t - V_t'$ at the opening and closing out positions of the portfolio, this is

$$e^{-rT}(h_{T,T}' Y_T - V_{T,T}') - \int_0^T e^{-rt}(Y_t + \sigma^2 Y_t^2 \Gamma_{t,T}')dh_{t,T}' - (h_{0,T}' Y_0 - V_{0,T}') = 0$$

This expression is the present value of the net cashflows of the replication that nullifies the debased option by notionally trading *Y*. This is entirely analogous to the expression for the debased portfolio cashflows of the binomial walk whereby the replicating asset remains debased with the slack variable which kills all net portfolio cashflow.

The Total Derivative of the Option Priced for the Demeaned Asset Y with Time Decay in the Market Asset S

Consideration is now given to the case whereby all of the components of the replicating portfolio have been debased with the artificial constraint $d\Pi_t = r\Pi_t dt$ whereby the parameter μ is surreptitiously[7] varied (debased) with the slack variable Δ (the asset *S*, the premium *V*, and the option delta *h*; to $S' = Y$, V', and h'), and then the demeaned replicating asset *Y* is remeaned (by removing the slack variable) to *S* whilst leaving the premium and delta debased [with the slack variable] at V' and h'. This corresponds to replicating the debased option with the debased delta by trading the market asset *S* rather than the demeaned asset *Y* for which the option is priced. This means that the debased premium primed with slack has time decay in the market asset rather than the demeamed asset for which it is accurate. The result is large changes in the debased premium as the compounded total slack for the transition to option maturity is amortized out of the premium with trading returns earned on *S*. The total change in the debased option premium for time decay in the market asset is

$$\frac{dV_0'}{dt_0} = rV_0' + (\mu + \tfrac{1}{2}\sigma^2 - r)S_0 h_0' + \sigma S_0 h_0' \frac{dB_0}{dt_0}$$

The addition change in the premium is a term that is proportional to the slack variable and the option delta, $\Delta Shdt$, which is the amount of slack amortized out of the debased premium for the differential change. It is also the term that is debased from the PDE of the accurate option premium by holding the portfolio flux fixed and flat at $d\Pi_t' = r\Pi_t' dt$. This term kills the market return earned on trading the market asset. Hence, the mispricing in the debased premium is subsidized from the trading returns earned on the market asset used to replicate, offset, and flatline the option. This means that the replicating portfolio remains primed throughout the replicating period [when rebalanced infinitely often] despite the replicating asset earning market return.

Proof: total derivative of debased option premium with time decay in the market asset *S*

The demeaned asset for which the debased premium is priced, the oxymoron risky bank account, has transition distribution

$$Y(T) \sim \log N\left(\log Y_0 + (r - \tfrac{1}{2}\sigma^2)(T - t_0),\ \sigma^2(T - t_0)\right)$$

and diffusion equation

$$\frac{dY_0}{dt} = rY_0 + \sigma Y_0 \frac{dB_0}{dt_0}$$

The debased premium is that which is priced (statistically) for the demeaned asset conditional on $Y(t_0) = S_0$

[7] and hence the portfolio is also surreptitiously primed (from Π to Π') with Δ; and remains primed regardless of whether or not the replicating asset *Y* is remeaned.

$$V_0' = V^{debased}(k,T;S_0,t_0) = \begin{cases} e^{-r(T-t_0)}\int_k^{\infty} (x-k)\, f_{Y(T)|Y(t_0)}(x \mid S_0)dx \\ e^{-r(T-t_0)}\int_0^{k} (k-x)\, f_{Y(T)|Y(t_0)}(x \mid S_0)dx \end{cases} = \begin{cases} S_0 \overline{F}_U\left(u_k' - \sigma\sqrt{T-t_0}\right) - ke^{-r(T-t_0)}\overline{F}_U\left(u_k'\right) \\ ke^{-r(T-t_0)}F_U\left(u_k'\right) - S_0 F_U\left(u_k' - \sigma\sqrt{T-t_0}\right) \end{cases}$$

and has the backward equation

$$-\frac{\partial V_0'}{\partial t_0} = \tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 V_0'}{\partial S_0^2} + rY_0 \frac{\partial V_0'}{\partial S_0} - rV_0'$$

as conditional on $Y(t_0) = S_0$ which, again, can be anything you like by conditioning since Markov processes have no memory [aside from where it was last conditioned]. Remeaning the demeaned asset from *Y* to *S* so that the option premium depends on the market asset rather than the demeaned asset (for which it is priced) for time decay (but keeping the premium and delta debased with the slack variable Δ), and expanding the total derivative of the option premium $V_0' = V_0'(t_0, S_0)$ in a power series and substituting in the backward equation

$$\begin{aligned} \frac{dV_0'}{dt_0} &= \frac{\partial V_0'}{\partial t_0}\frac{dt_0}{dt_0} + \frac{\partial V_0'}{\partial S_0}\frac{dS_0}{dt_0} + \tfrac{1}{2}\frac{\partial^2 V_0'}{\partial S_0^2}\frac{(dS_0)^2}{dt_0} \\ &= \frac{\partial V_0'}{\partial t_0} + \frac{\partial V_0'}{\partial S_0}\frac{dS_0}{dt_0} + \tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 V_0'}{\partial S_0^2} \\ &= -\left(\tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 V_0'}{\partial S_0^2} + rS_0\frac{\partial V_0'}{\partial S_0} - rV_0'\right) + \frac{\partial V_0'}{\partial S_0}\frac{dS_0}{dt_0} + \tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 V_0'}{\partial S_0^2} \\ &= rV_0' - rS_0\frac{\partial V_0'}{\partial S_0} + \frac{\partial V_0'}{\partial S_0}\frac{dS_0}{dt_0} \end{aligned}$$

and given that the diffusion equation for the risky market asset *S* for which the option depends is $dS_0 = (\mu + \tfrac{1}{2}\sigma^2)S_0 dt_0 + \sigma S_0 dB_0$, rather than $dY_0 = rY_0 dt_0 + \sigma Y_0 dB_0$ since the replicating asset has been re-meaned, so that $dS_0 - rS_0 dt_0 = (\mu + \tfrac{1}{2}\sigma^2 - r)S_0 dt_0 + \sigma S_0 dB_0$ this can be written as

$$\begin{aligned} \frac{dV_0'}{dt_0} &= rV_0' + \frac{\partial V_0'}{\partial S_0}\left(\frac{dS_0}{dt_0} - rS_0\right) \\ &= rV_0' + \frac{\partial V_0'}{\partial S_0}\left((\mu + \tfrac{1}{2}\sigma^2 - r)S_0 + \sigma S_0\frac{dB_0}{dt_0}\right) \\ &= rV_0' + (\mu + \tfrac{1}{2}\sigma^2 - r)S_0\frac{\partial V_0'}{\partial S_0} + \sigma S_t\frac{\partial V_0'}{\partial S_0}\frac{dB_0}{dt_0} \end{aligned}$$

i.e.,

$$\frac{dV_0'}{dt_0} = rV_0' + (\mu + \tfrac{1}{2}\sigma^2 - r)S_0 h_0' + \sigma S_0 h_0'\frac{dB_0}{dt_0}$$

This expression has the additional term ($\Delta S_0 h_0'$), which is proportional to the slack variable $\Delta = \mu + \tfrac{1}{2}\sigma^2 - r$ and the option delta h_0', due to time decay in *S* rather than *Y* for which the premium is priced. This additional term amortizes the compounded slack in the debased premium [i.e., removes $\Delta h_0 S_0 dt_0$ from the debased premium over each infinitesimal transition] as it tries to catch up with the value of the accurate premium as option expiry approaches and the derivative reverts to an optional spot transaction and the two premiums move into parity [all the slack removed] and the option ceases to be a derivative [the deferral period expires and the derivate reverts to the spot transaction]. In the context of the replicating portfolio whereby the asset is remeaned to *S* whilst keeping the premium and deltas debased with Δ, this term

$$\Delta S_0 h_0' = (\mu + \tfrac{1}{2}\sigma^2 - r)S_0\frac{\partial V_0'}{\partial S_0}$$

kills the market return earned on the market asset traded to replicate and offset the option, and is the term that is debased from the PDE of the accurate premium by debasing with $d\Pi_t = r\Pi_t dt$ [and inadvertently priming the

portfolio, asset, premium, and deltas with slack Δ]. The result is a portfolio return that is kept debased, despite remeaning the replicating asset, and kept void of market return and the benefits of trading an economic asset. The result is no option and no return. The artificial constraint $d\Pi_t = r\Pi_t dt$ placed on the replicating portfolio therefore surreptitiously debases derivatives and implies derivatives have no economic value and are consequently worthless redundant securities [in all circumstances, including the one in which we live].

The Return Components of the Debased Option Replicating Portfolio that Replicates with the Market Asset S

The debased portfolio whereby the asset is remeaned while leaving the option premium and delta debased with slack is

$$\Pi_t^{debased} = h_t^{debased} S_t - V_t^{debased}$$

and it is proved by three methods (rigorously by two) that the result is the same as though a demeaned asset was traded to replicate the debased premium priced for the demeaned asset. This occurs by the mispricing in the premium sinking the market returns of the remeaned asset.

$$d\Pi_t' = r\Pi_t' dt + (S_t + dS_t)dh_t$$

which takes the market return of the market asset and transfers it into the option premium so that it does not accrue to the portfolio but instead goes to biasing, and hence subsidizing, a mispriced premium; the premium essentially becomes a sink [from the mispricing] for the trading returns [a source].

Proofs: debased portfolio returns with time decay in the market asset *S* (the remeaned demeaned asset)

Proof 1: heuristic proof

Surmising heuristically that the market asset is going to earn market return regardless of how the premium is priced or any constraint placed on the dynamics of the portfolio. For a replicating portfolio with dynamics fixed flat at cash $d\Pi_t^{debased} = r\Pi_t^{debased} dt$ that trades the market asset, the market asset is going to earn $\mu + \frac{1}{2}\sigma^2$ anyway so

$$\begin{aligned} d\Pi_t^{debased} &= r\Pi_t^{debased} dt \\ &= r\left(h_t^{debased} S_t - V_t^{debased}\right)dt \\ &= \left(rh_t^{debased} S_t - rV_t^{debased}\right)dt \\ &= \left((\mu + \tfrac{1}{2}\sigma^2)h_t^{debased} S_t - rV_t^{debased} + rh_t^{debased} S_t - (\mu + \tfrac{1}{2}\sigma^2)h_t^{debased} S_t\right)dt \\ &= (\mu + \tfrac{1}{2}\sigma^2)h_t^{debased} S_t dt - \left(rV_t^{debased} + (\mu + \tfrac{1}{2}\sigma^2 - r)h_t^{debased} S_t\right)dt \end{aligned}$$

which shows that the portfolio change of the debased option replicated by the market asset with the debased option delta is kept fixed flat at the cash rate by the market return on the market asset being killed by the mispricing in the debased premium.

The market asset earns the market rate while the short option incurs a large loss made up of two parts **(1)** the cash rate in proportion to the option ($rV_t^{debased}$) and **(2)** an additional loss in proportion to the excess market rate of return, the debased option delta, and the spot value of the market asset ($\Delta h_t^{debased} S_t$). This additional loss on the short option, resulting in a loss on the short option in excess of both the cash rate and the market return (since $h_t^{debased} S_t > V_t^{debased}$), kills the market return earned on the traded asset keeping the return on the portfolio fixed flat at cash [rather than with convex flux in market return] and debased from the benefits of trading an economic asset linked to economic growth.

Proof 2: total derivative with time decay in the market asset *S*

The replicating portfolio for the portfolio with everything debased whereby the traded asset is then remeaned, and hence *S* rather than *Y* for which the debased premium is accurate, is the circumstance where the option priced for *Y* is replicated by trading *S* with the deltas of the option accurate for *Y* whereby the slack variable is removed from the replicating asset but left in the premium and deltas. The return on the offsetting asset is the market rate and hence the time decay in the option premium depends on the asset which returns the market rate, for the premium which is priced accurately for a different asset which returns only the cash rate. The result is a

portfolio kept debased and null of the benefits of markets, despite trading the market asset *S* to offset and flatline the option.

The total derivative of the option priced for the demeaned asset *Y* but with time decay in the market asset S, rather than the demeaned asset *Y*, is

$$\frac{dV_t'}{dt} = rV_t' - rS_t\frac{\partial V_t'}{\partial S_t} + \frac{\partial V_t'}{\partial S_t}\frac{dS_t}{dt}$$

This is where the option is priced for $S' = Y$ but depends on *S* rather than *Y* for time decay.

The portfolio whereby all items have been debased with the slack variable, and then the slack variable is removed from the replicating asset, so that the debased premium priced for *Y* is replicated by trading *S* is

$$\Pi_t^{debased} = h_t^{debased}S_t - V_t^{debased}$$

i.e., $\Pi_t' = h_t'S_t - V_t'$ rather than $\Pi_t' = h_t'S_t' - V_t' = h_t'Y_t - V_t'$. Differentiating totally and substituting the total derivative of the option premium into the total portfolio derivative

$$\frac{d\Pi_{t,T}'}{dt} = h_{t,T}'\frac{dS_t}{dt} - \left(rV_t' - rS_t\frac{\partial V_t'}{\partial S_t} + \frac{\partial V_t'}{\partial S_t}\frac{dS_t}{dt}\right) + (S_t + dS_t)\frac{dh_{t,T}'}{dt} \qquad \text{where} \quad h_t' = \frac{\partial V_t'}{\partial S_t}$$

$$= rh_t'S_t - rV_t' + (S_t + dS_t)\frac{dh_{t,T}'}{dt}$$

$$= r\Pi_t' + (S_t + dS_t)\frac{dh_{t,T}'}{dt}$$

after cancelling the replicating delta in the market asset. Net of the update, this is $d\Pi_{t,T}' = r\Pi_t'dt$.

To elucidate what is going on, namely subsidizations between asset and premium, the total derivative of the option written in terms of standard Brownian motion was proved to be $dV_t' = rV_t'dt + (\mu + \frac{1}{2}\sigma^2 - r)S_th_t'dt + \sigma S_th_t'dB_t$ and the diffusion equation of the replicating market asset is $dS_t = (\mu + \frac{1}{2}\sigma^2)S_tdt + \sigma S_tdB_t$ so $h_{t,T}'dS_t$ can be written as $h_t'dS_t = (\mu + \frac{1}{2}\sigma^2)h_t'S_tdt + \sigma h_t'S_tdB_t$. The total change in the replicating portfolio can be expressed as

$$\frac{d\Pi_{t,T}'}{dt} = \left((\mu + \tfrac{1}{2}\sigma^2)h_t'S_t + \sigma h_t'S_t\frac{dB_t}{dt}\right) - \left(rV_t' + (\mu + \tfrac{1}{2}\sigma^2 - r)S_t\frac{\partial V_t'}{\partial S_t} + \sigma S_t\frac{\partial V_t'}{\partial S_t}\frac{dB_t}{dt}\right) + (S_t + dS_t)\frac{dh_{t,T}'}{dt} \quad \text{where } h_t' = \frac{\partial V_t'}{\partial S_t}$$

$$= (\mu + \tfrac{1}{2}\sigma^2)h_t'S_t - \left(rV_t' + (\mu + \tfrac{1}{2}\sigma^2 - r)S_th_t'\right) + (S_t + dS_t)\frac{dh_{t,T}'}{dt}$$

$$= rh_t'S_t - rV_t' + (S_t + dS_t)\frac{dh_{t,T}'}{dt}$$

$$= r\Pi_t' + (S_t + dS_t)\frac{dh_{t,T}'}{dt}$$

which shows clearly, net of the offsetting variation, that

$$E\left(h_t'dS_t \mid S_t\right) = (\mu + \tfrac{1}{2}\sigma^2)h_t'S_tdt \qquad \text{and} \qquad \begin{aligned} E\left(dV_t' \mid S_t\right) &= rV_t'dt + (\mu + \tfrac{1}{2}\sigma^2 - r)S_th_t'dt \\ &= rV_t'dt + \Delta S_th_t'dt \end{aligned}$$

so that the position in the market asset earns the market rate of return $\mu + \frac{1}{2}\sigma^2$ whilst the debased option undergoes a large time value change and the short position incurs $\left(rV_t' + (\mu + \frac{1}{2}\sigma^2 - r)S_th_t'\right)/V_t'$ which exceeds the market rate of return. The additional term in the slack variable for the change in the option premium then kills the market return earned on the market asset debasing the portfolio to that for which trades the demeaned asset in order to replicate the debased premium, i.e., a debased portfolio that is void of [convex flux in] market return [despite the market asset earning market return]. There is then a cross subsidization between the debased premium and the market asset whereby the return earned on the market asset subsidizes, and is sunk by, the mispricing in the debased premium. This is what return-neutral valuation calls a *"return to the real P world"*. It is, instead, just the case whereby the replicating asset is remeaned (by removing the slack variable from *Y*) whilst

keeping the premium and deltas debased with the slack variable Δ. Rather than being **risk free in every world**, it is **return free in any circumstance** (whatever the remeaned asset earns). The returns earned on the remeaned asset then subsidize the mispricing in the premium debased with slack keeping the portfolio return debased and fixed flat at r and void of the benefits of trading and participating in the economic activity of the financial markets. This debases derivatives and turns them into redundant and useless securities of no economic value which, ironically, then have to be flatlined in order to be subsidized. It also implies, further, that the securities of the spot markets also have no economic value in that the economic activity of these assets over the deferral period of the derivative is worthless implying further that, not only are the derivative markets markets for redundant securities, but so too are the spot markets.

The only way to achieve the goals of return-neutral valuation is to shut the derivative markets down, then all option payoff profiles and all net cashflows are zeroed, without having to do a hellovalot of trading for [ironically] absolutely nothing except to kill the derivatives market. This is not possible otherwise and the goals that return-neutral valuation inadvertently set out to achieve are always a failure.

Proof 3: deconvolution integral by manipulations of the slack variable

This proof corresponds one-for-one with that of PART I for discrete time and a binomial basis function. See page [insert here]. Since it is the market asset that is traded and used to replicate the option, the slack variable is removed from the basis function of the forward transition of the components of the portfolio so that the basis components are with respect to the distribution with the correct exponential return μ [and hence free from the slack variable Δ]. The time decay in the premium [that remains debased with slack Δ] then depends on the market asset that is free from slack. The slack variable remains in the option premium and option delta and these remain accurate only for the demeaned asset that is demeaned and debased.

The mathematics of the proof of the resulting portfolio return components reintroduces the slack variable with a null additive term ($\Delta-\Delta=0$) for $\mu=\mu-\Delta+\Delta=r-\frac{1}{2}\sigma^2+\Delta$ into the basis components X, separates Δ out [whilst leaving the drift debased at $r-\frac{1}{2}\sigma^2$] in a series expansion after a substitution and change of variable, which then leaves the basis components debased [and primed X'] so that the recurrences of the debased option premium and delta can then be utilized and substituted into the dynamics to complete the proof. This proof is complicated, and is essentially saying the same thing as the other two proofs, but is elucidating and corresponds one-for-one with the proof of the binomial replicating portfolio of PART I (which is a shame that it is not quite so elegant for debasing by varying step ratios as it is by varying probabilities since varying p is nonsense with regard to sound probability theory [hence, for example, negative probabilities]).

Holding delta constant and differentiating for the portfolio returns, taking expectations, and then separating out the slack variable

$$
\begin{aligned}
d\Pi_t^{\,debased} &= E\left(d\Pi_t^{\,debased} \mid \Pi_t^{\,debased}\right) \\
&= E\left(\Pi^{debased}(k,T;S_{t+dt},t+dt)-\Pi^{debased}(k,T;S_t,t)\right) \\
&= E\left(\left(h_t^{\,debased}S_{t+dt}-V^{debased}(k,T;S_{t+dt},t+dt)\right)-\left(h_t^{\,debased}S_t-V^{debased}(k,T;S_t,t)\right)\right) \\
&= E\left(\left(h_t^{\,debased}S_{t+dt}-h_t^{\,debased}S_t\right)-\left(V^{debased}(k,T;S_{t+dt},t+dt)-V^{debased}(k,T;S_t,t)\right)\right) \\
&= \int_0^\infty f_X(x)\left(\left(h_t^{\,debased}S_t\times x-h_t^{\,debased}S_t\right)-\left(V^{debased}(k,T;S_t\times x,t+dt)-V^{debased}(k,T;S_t,t)\right)\right)dx \\
&= \left(\int_0^\infty xf_X(x)dx-1\right)h_t^{\,debased}S_t-\left(\int_0^\infty f_X(x)V^{debased}(k,T;S_t\times x,t+dt)dx-V^{debased}(k,T;S_t,t)\right) \\
&= \left(e^{(\mu+\frac{1}{2}\sigma^2)dt}-1\right)h_t^{\,debased}S_t-\left(e^{rdt}\int_0^\infty f_X(x)e^{-rdt}V^{debased}(k,T;S_t\times x,t+dt)dx-V^{debased}(k,T;S_t,t)\right)
\end{aligned}
$$

where the expectations are with respect to the market asset S rather than the demeaned asset Y i.e., with respect to the basis density X for the asset whereby the slack variable has been removed [but left in the premium and delta] so that the replicating asset [and asset underlying the derivative] has been remeaned while leaving the option premium and its delta debased with the slack variable

$$
f_X(x)=\frac{1}{x\sigma\sqrt{2\pi\,dt}}e^{-\frac{1}{2}\frac{(\log x-\mu dt)^2}{\sigma^2 dt}} \qquad \text{rather than} \qquad f_{X'}(x)=\frac{1}{x\sigma\sqrt{2\pi\,dt}}e^{-\frac{1}{2}\frac{[\log x-(r-\frac{1}{2}\sigma^2)dt]^2}{\sigma^2 dt}}
$$

This is the so-called "return to the real world" from an alternative world that does not exist that "has no risk" but is true in every world (which is return-neutral nonsense), which is, in sensible words, remeaning the demeaned

asset to *S* whilst keeping the premium and deltas debased. What return-neutral valuation really means by "it is true in every world" is that the portfolio earns nothing no matter what the market asset earns. This is, of course, because the premium which remains debased with Δ kills it all.

Note that the premium is only a bookvalue between writing the option and closing out the portfolio so that the mispricing in the initial premium is responsible for killing the majority of the return that is then debased from the portfolio over the replication period. The remainder [of the bias] is from less return earned from slightly smaller positions because of smaller deltas, which otherwise carry the return for the market.

The recursion in the premium is with respect to the demeaned asset but it is with respect to the basis components *X* of the market asset rather than the debased basis components of the demeaned asset for which applies to the debased premium, i.e., time decay in the original basis components *X* of *S* rather than the debased basis components *X′* of *Y* and hence time decay in the market asset *S* rather than the demeaned asset *Y* [despite still being priced for *Y* since the slack has been left in *V*]. There is not a recursion for this situation so the remeaned basis components have to be again reprimed with Δ. So writing the basis density in terms of the debased distribution and the slack variable with the null additive term $\Delta-\Delta=0$ to write the undebased exponential drift as $\mu=\mu+\Delta-\Delta=r-\frac{1}{2}\sigma^2+\Delta$ for

$$f_X(x)=\frac{1}{x\sigma\sqrt{2\pi\,dt}}e^{-\frac{1}{2}\frac{(\log x-\mu dt)^2}{\sigma^2 dt}}=\frac{1}{x\sigma\sqrt{2\pi\,dt}}e^{-\frac{1}{2}\frac{[\log x-(r-\frac{1}{2}\sigma^2+\Delta)dt]^2}{\sigma^2 dt}}$$

where one Δ is reintroduced into the basis density to reprime it so that the recursions for the debased premium can be utilized to get the returns, and the other Δ is expanded out in a series for *V*;

$$d\Pi_t^{\,debased}=\left(e^{(\mu+\frac{1}{2}\sigma^2)dt}-1\right)h_t^{\,debased}S_t-\left(e^{rdt}\int_0^\infty\frac{1}{x\sigma\sqrt{2\pi\,dt}}e^{-\frac{1}{2}\frac{[\log x-(r-\frac{1}{2}\sigma^2+\Delta)dt]^2}{\sigma^2 dt}}V^{debased}(k,T;S_t\times x,t+dt)dx-V^{debased}(k,T;S_t,t)\right)$$

making a change of variable

$$\frac{\log x-(r-\frac{1}{2}\sigma^2+\Delta)dt}{2\sigma^2}=\frac{\log y-(r-\frac{1}{2}\sigma^2)dt}{2\sigma^2}$$

$$\log x-\Delta dt=\log y$$

$$x=ye^{\Delta dt}$$

$$dx=e^{\Delta dt}dy$$

$$d\Pi_t^{\,debased}=\left(e^{(\mu+\frac{1}{2}\sigma^2)dt}-1\right)h_t^{\,debased}S_t-\left(e^{rdt}\int_0^\infty\frac{1}{y\sigma\sqrt{2\pi\,dt}}e^{-\frac{1}{2}\frac{[\log y-(r-\frac{1}{2}\sigma^2)dt]^2}{\sigma^2 dt}}V^{debased}(k,T;S_t\times ye^{\Delta dt},t+dt)dx-V^{debased}(k,T;S_t,t)\right)$$

expanding the premium in a power series around $\log S_t y$ in order to separate out the slack variable that was introduced into the argument for the backward variable of the premium [when the basis components were reprimed]

$$
\begin{aligned}
d\Pi_t^{debased} &= \left(e^{(\mu+\frac{1}{2}\sigma^2)dt}-1\right)h_t^{debased}S_t + o(dt) \\
&\quad -\left(e^{rdt}\int_0^{\infty}\frac{1}{y\sigma\sqrt{2\pi\,dt}}e^{-\frac{1}{2}\frac{[\log y-(r-\frac{1}{2}\sigma^2)dt]^2}{\sigma^2 dt}}\left\{1+\Delta dt\frac{\partial}{\partial\log S_t y}\right\}V^{debased}(k,T;S_t\times y,t+dt)dx - V^{debased}(k,T;S_t,t)\right) \\
&= \left(e^{(\mu+\frac{1}{2}\sigma^2)dt}-1\right)h_t^{debased}S_t + o(dt) \\
&\quad -\left(e^{rdt}\int_0^{\infty}\frac{1}{y\sigma\sqrt{2\pi\,dt}}e^{-\frac{1}{2}\frac{[\log x-(r-\frac{1}{2}\sigma^2)dt]^2}{\sigma^2 dt}}\left\{1+\Delta dt\,S_t y\frac{\partial}{\partial S_t y}\right\}V^{debased}(k,T;S_t\times y,t+dt)dx - V^{debased}(k,T;S_t,t)\right) \\
&= \left(e^{(\mu+\frac{1}{2}\sigma^2)dt}-1\right)h_t^{debased}S_t - \Delta S_t dt\,e^{rdt}\int_0^{\infty}\frac{1}{y\sigma\sqrt{2\pi\,dt}}e^{-\frac{1}{2}\frac{[\log x-(r-\frac{1}{2}\sigma^2)dt]^2}{\sigma^2 dt}}y\frac{\partial}{\partial S_t y}V^{debased}(k,T;S_t\times y,t+dt)dx \\
&\quad -\left(e^{rdt}\int_0^{\infty}\frac{1}{y\sigma\sqrt{2\pi\,dt}}e^{-\frac{1}{2}\frac{[\log x-(r-\frac{1}{2}\sigma^2)dt]^2}{\sigma^2 dt}}V^{debased}(k,T;S_t\times y,t+dt)dx - V^{debased}(k,T;S_t,t)\right) + o(dt) \\
&= \left(e^{(\mu+\frac{1}{2}\sigma^2)dt}-1\right)h_t^{debased}S_t - \Delta S_t dt\,e^{rdt}\int_0^{\infty}\frac{1}{y\sigma\sqrt{2\pi\,dt}}e^{-\frac{1}{2}\frac{[\log x-(r-\frac{1}{2}\sigma^2)dt]^2}{\sigma^2 dt}}y\times h^{debased}(k,T;S_t\times y,t+dt)dx \\
&\quad -\left(e^{rdt}\int_0^{\infty}\frac{1}{y\sigma\sqrt{2\pi\,dt}}e^{-\frac{1}{2}\frac{[\log x-(r-\frac{1}{2}\sigma^2)dt]^2}{\sigma^2 dt}}V^{debased}(k,T;S_t\times y,t+dt)dx - V^{debased}(k,T;S_t,t)\right) + o(dt)
\end{aligned}
$$

where the basis density functions are now primed with slack so that the recurrences for the debased option premium and option delta can be substituted. The backward recurrence of the option premium to be substituted is $\int_0^{\infty} f_{X'}(x)e^{-rdt}V^{debased}(k,T;S_t\times x,t+dt)dx = V^{debased}(k,T;S_t,t)$ and that for the option delta is $\int_0^{\infty} y\times f_{X'}(x)e^{-rdt}h^{debased}(k,T;S_t\times x,t+dt)dx = h^{debased}(k,T;S_t,t)$ where the basis density functions in the two recursions are now primed with slack $\Delta=\mu+\frac{1}{2}\sigma^2-r$ for the basis variable X' which then apply to the premium and deltas of the portfolio [V' and h'] which have remained debased after the asset is remeaned and hence change with respect to X rather than X'

$$d\Pi_t^{debased} = \left(e^{(\mu+\frac{1}{2}\sigma^2)dt}-1\right)h_t^{debased}S_t - \left(e^{rdt}-1\right)V^{debased}(k,T;S_t,t) - (\mu+\tfrac{1}{2}\sigma^2-r)S_t h_t^{debased}dt + o(dt)$$

which is the market rate of return on the market asset and the cash rate on the short option, plus an additional loss ($\Delta Sh'$) on the short option in proportion to the market spot price, the option delta, and the slack variable. This additional loss of the short option means that it incurs more than the market rate of return (since $hS > V$) and consequently kills all of the market return earned on the offsetting asset used to kill the payoff profile of the option. Separating out non-$o(dt)$ terms for

$$d\Pi_t^{debased} = \left((\mu+\tfrac{1}{2}\sigma^2)dt\right)h_t^{debased}S_t - \left(rdt\right)V^{debased}(k,T;S_t,t) - (\mu+\tfrac{1}{2}\sigma^2-r)S_t h_t^{debased}dt + o(dt)$$

and dividing through by dt

$$\frac{d\Pi_t^{debased}}{dt} = (\mu+\tfrac{1}{2}\sigma^2)h_t^{debased}S_t - rV^{debased}(k,T;S_t,t) - (\mu+\tfrac{1}{2}\sigma^2-r)S_t h_t^{debased} + \frac{o(dt)}{dt}$$

Taking limits by letting $dt \to 0$

$$\frac{d\Pi_t^{debased}}{dt} = (\mu+\tfrac{1}{2}\sigma^2)h_t^{debased}S_t - \left(rV_t^{debased} + (\mu+\tfrac{1}{2}\sigma^2-r)S_t h_t^{debased}\right)$$

as surmised heuristically.

Return-neutral valuation argued wrongly that people would become infinitely rich if prices and portfolios were not debased, since they did not know what the correct mathematics was, which this thesis has now provided. The assertion that people would become infinitely rich from option prices that are not debased is wrong and has seen to be wrong empirically ever since 1973 when they first asserted this. Prices are not debased and never

have been yet people have not had their very own "perpetual money making machines" of infinite profits from option prices that are not debased (as some return-neutral theorists asserted [and failed PhDs with]). If theory worked properly and deltas could be maintained continuously, then if option prices and portfolios were not debased, and were instead correctly priced, people would earn the economic value of the derivative that they are passing up by replicating and nullifying their option position [and taking up the pecuniary interest in the economic asset over the deferral period of the derivative]. This is consistent with earnings derived from trading any economic asset within a modern economy, in this case, by providing a financial service by making a market in the derivative. In GDP terms, this is the value added to the economy by the provision of the economic good or service, in this case, a deferred transaction of the spot markets [which is then measureable by the value added method of calculating GDP]. By debasing derivatives, it is implied that derivatives have no economic value and add nothing to the economy in which options are traded, i.e., are redundant securities of no economic benefit and the financial service of making the market in the derivative is worthless. Options are not redundant worthless securities and do have economic benefit [and contribute to GDP] and the debased price of Black and Scholes is wrong. The correct price is the statistically accurate price, which recognizes market return μ, as proposed and argued vehemently for by this thesis with sound [and original] mathematical logic and proof. Yeh!

■

The NPV of Cashflows for the Debased Option Replicating Portfolio that Replicates with the Market Asset S

Considering the replicating portfolio where every item within the portfolio are debased with slack, the premium, the delta, and the replicating asset and hence the cashflows are for the *"alternative Q world that does not exists for derivative pricing"*. The total change in the debased portfolio Π_t' is

$$d\Pi_t' = r\Pi_t' dt + (Y_t + dY_t)dh_t' \qquad \Pi_t' = h_t' Y_t - V_t'$$

where $V_t' = e^{-r(T-t)}E\left(\max\left(Y(T)-k,0\right)|Y(t)=Y_t\right)$ and $V_T' = E\left(\max\left(Y(T)-k,0\right)|Y(T)=Y_T\right)$ for a call option and $V_t' = e^{-r(T-t)}E\left(\max\left(k-Y(T),0\right)|Y(t)=Y_t\right)$ and $V_T' = E\left(\max\left(k-Y(T),0\right)|Y(T)=Y_T\right)$. Now remeaning the replicating asset from *Y* to *S* by removing the slack variable from the demeaned asset whilst keeping the premium and deltas debased with the slack variable (ignoring replicating problems) for the so-called *"return to the real world P"* whereby the market returns of the market asset are sunk by the mispriced option keeping the portfolio null of the benefits of participating in the markets and are kept fixed flat at nothing. The total change in the portfolio that remains debased [despite trading the market asset] is

$$d\Pi_t' = r\Pi_t' dt + (S_t + dS_t)dh_t'$$

or

$$d\Pi_t' = r\Pi_t' dt + (S_t + \sigma^2 S_t^2 \Gamma_t')dh_t'$$

net of further $o(dt)$ variation in the rebalance, where $V_t' = e^{-r(T-t)}E\left(\max\left(Y(T)-k,0\right)|Y(t)=S_t\right)$ and $V_T' = E\left(\max\left(Y(T)-k,0\right)|Y(T)=S_T\right)$ for a call option and $V_t' = e^{-r(T-t)}E\left(\max\left(k-Y(T),0\right)|Y(t)=S_t\right)$ and $V_T' = E\left(\max\left(k-Y(T),0\right)|Y(T)=S_T\right)$ for a put option. The delta update is then a buy or a sell in the market asset *S*, rather than the demeaned asset *Y*, and time decay in the debased premium is with respect to *S* rather than *Y*. The differential equation of cashflows is then

$$\frac{d\Pi_t'}{dt} - r\Pi_t' = (S_t + \sigma^2 S_t^2 \Gamma_t')\frac{dh_t'}{dt}$$

Differentially factorizing with the integrating factor $IF(t) = e^{-rt}$

$$\frac{d}{dt}\left(e^{-rt}\Pi_t'\right) = e^{-rt}(S_t + \sigma^2 S_t^2 \Gamma_t')\frac{dh_t'}{dt}$$

and integrating over the replication period to option expiration at time *T*

$$\int_0^T \frac{d}{dt}\left(e^{-rt}\Pi_t'\right)dt = \int_0^T e^{-rt}(S_t + \sigma^2 S_t^2 \Gamma_t')\frac{dh_t'}{dt}dt$$

i.e.,

$$e^{-rT}\Pi_T' - \int_0^T e^{-rt}(S_t + \sigma^2 S_t^2 \Gamma_t')dh_t' - \Pi_0' = 0$$

which is the NPV of the cashflows of the replicating portfolio for that which replicates the debased premium by trading the market asset *S*. In terms of the portfolio contents $\Pi_t' = h_t' S_t - V_t'$ at the opening and closing out positions of the portfolio

$$e^{-rT}(h_{T,T}' S_T - V_{T,T}') - \int_0^T e^{-rt}(S_t + \sigma^2 S_t^2 \Gamma_{t,T}')dh_{t,T}' - (h_{0,T}' S_0 - V_{0,T}') = 0$$

This expression gives the present value of the net cashflows of the replication that nullifies the debased option by trading *S*. Aside from plenty of o(*dt*) variation, this is entirely analogous to the expression for the portfolio cashflows of the binomial walk whereby the replicating asset is remeaned by removing the slack variable, whilst keeping the premiums and deltas debased, which [similarly to that which notionally trades *Y*] kills all net portfolio cashflow since the mispricing in the premium which remains debased kills all of the return that the remeaned asset earns [slack in V' sinks the source from *S*]. No option and no returns and a hellovalot of trading. Although zeroing all portfolio cashflows and flatlining options is a pointless exercise, particularly if you like options, it is not possible in actual markets anyway [entirely asides from the fact that it is fanciful to assume that all market assets are exponential diffusion which are the only diffusions for which exponential sources and sinks apply].

The formula for net portfolio cashflows shows that whenever the portfolio is primed so that the infinitesimal portfolio return is fixed flat and debased at *r*, the net cashflow is zero regardless of the path of the underlying asset and the resulting path of the debased option premium and its delta since the convex flux in market return has been killed by the mispriced option as a consequence of holding $d\Pi_t = r\Pi_t dt$ [which surreptitiously primes the premium and portfolio with slack]. The present value of

(1) the closing out proceeds $e^{-rT}\Pi_T'$

(2) less the total present value of the delta updates $\int_0^T e^{-rt}(S_t + \sigma^2 S_t^2 \Gamma_t')dh_t'$

(3) less the initial cost of establishing the portfolio Π_0'

is zero for any path. No option and no return and a hellovalot of trading; there is heavy trading in the underlying market asset, many times the value of the initial option premium, and this asset grows at a rate greater than the cash rate of return. So where has the return from all of this trading gone? Like that for the binomial walk, the market returns have been zeroed by pricing the option for the demeaned asset and the values of the resulting mispriced option premium V' then sink the returns of the market asset *S*, whatever those returns are. This is what return-neutral valuation calls *"true in every world"* which really means that the portfolio *earns nothing in every circumstance*. If all of the expected market trading returns of the market asset in the replicating portfolio [expected value of the convex market flux] are brought forward and moved into the statistically unbiased and accurate premium from an accurate replication, then the resulting premium is very close to that which has been primed and debased with slack;

- In the case of a call option where delta is positive, these market trading returns, which are earned by flatlining the option with the replicating asset with a net long position, are used to subsidize writing a cheap underpriced call;
- In the case of a put option where delta is negative, the market trading losses on trading short the market asset, which are incurred by flatlining the option with a net short position, is financed by writing an overpriced put.

Hence, whenever the drift of the market asset underlying the derivative is greater than the cash rate, **calls are underpriced** and **puts are overpriced** by the debased formulae of Black and Scholes. In terms of statistical bias, for market returns in excess of cash, options priced at the debased premium will result in the **call option** price being **statistically biased downwards** [statistically underpriced for the long transaction of the derivative] and the **put option** price being **statistically biased upwards** [statistically overpriced for the short transaction of the derivative]. The reverse if the drift of the market asset is less than the cash rate; for example, in economic recession. As was shown for the binomial asset and replicating portfolio, the source of the statistical bias arises from debasing the portfolio with $d\Pi_t = r\Pi_t dt$ and inadvertently introducing the slack variable Δ into the exponential mean which debases the premium and then biases the expectation so that it is inaccurate for the market asset for which the derivative expires. This is most evident in the inaccurate probabilities which apply to *Y* rather than *S*. The replicating arguments for debased portfolios then attempt to subsidize this mispricing by killing the option in order to justify the statistically biased and systematically mispriced premium. This is the pointless

and worthless circularity that has inadvertently been the justification to debase and systematically misprice derivatives.

Flatlined Option Payoff Profiles [killed options]

Replicating a debased option, with either S or notionally with $S' = Y$, flatlines the payoff profile of the option, i.e., turns the payoff profile of an option into a flatline **[similar to a life support system when a patient dies]**. What is meant by this is shown clearly diagrammatically for the continuous exponential Brownian motion [which has a continuous diffusion sample space for S]

Long call **replicated long debased call**



Long put **replicated long debased put**



Short call **replicated short debased call**



Short put **replicated short debased put**



This is what return-neutral valuation means by *"an alternative Q world that does not exist"* [when replicated with Y] and *"being true in every world, including the one in which we live P"* [when replicated with S], which actually means *"it earns nothing in every circumstance [whatever the replicating asset earns] for a flatlined and redundant security"*. If you want an option, you must not replicate and kill it. If all options must be replicated, in order to be subsidized [to subsidize the debased premium], as return-neutral says, then option payoff profiles should not be taught to students because they are then irrelevant. Further, the option should not be written in the first place and the option markets are better off closed down. Does it make sense to say that the surface of Mercury has no risk with respect to living organisms? There is no risk in a flatlined option because it is dead [if

it worked[8], which it doesn't]. **Do you really want this for options**? The alternative is to recognize the **return of the underlying spot markets** and the **economic value of derivatives**, and then try to price them as accurately as possible. The sensible PDE for options is then $V_t = rV - (\mu + \frac{1}{2}\sigma^2) S V_S - \frac{1}{2}\sigma^2 S^2 V_{SS}$ [and that for futures is this without the gamma $V_t = rV - (\mu + \frac{1}{2}\sigma^2) S V_S$] which is that for the premium accurate for the market asset for which the spot transaction is being deferred by the derivative.

Explaining the Statistical Bias in the Debased Premium in the Context of an Option Replicating Portfolio

Comparison of Replicating Portfolios and the NPV of Cashflows

The portfolio cashflow equations are the solutions to the differential equations of the infinitesimal cashflows inclusive of the delta-updates

$$e^{-rT}\Pi_T - \int_0^T e^{-rt}(S_t + \sigma^2 S_t^2 \Gamma_t) dh_t - \Pi_0 = (\mu + \tfrac{1}{2}\sigma^2 - r)\int_0^T e^{-rt} S_t h_t dt$$

for **a portfolio free to earn return with convex market flux** $d\Pi_{t,T} = (\mu + \frac{1}{2}\sigma^2) S_t h_{t,T} dt - rV_{t,T} dt$; and

$$e^{-rT}\Pi_T^{debased} - \int_0^T e^{-rt}(S_t + \sigma^2 S_t^2 \Gamma_t^{debased}) dh_t^{debased} - \Pi_0^{debased} = 0$$

for **a debased portfolio fixed flat at cash** $d\Pi_{t,T}^{debased} = r\Pi_{t,T}^{debased} dt$. Observing the difference between the two formulae for NPV of cashflows, that of the right hand side, one representing the market return earned on the period-to-period position in the market asset and the other zero. The portfolio assets are still earning return regardless and it is the mispricing in the debased premium that kills the return on assets to leave a cashflow equation void of market return. Since, apart from expiry when the two portfolios are equivalent for zero transition [and whence the forward variable equals the backward variable and the backward variables of *S* and *Y* are again in parity by conditioning], the only transaction in the variable *V* for the forward transition of the replication period is the initial transaction when the option is bought or sold and the premium is exchanged. The only other source is returns on the traded market asset. Therefore, the return on assets that is killed by the mispriced premium must be almost entirely contained in the initial premium. Hence, the statistical inaccuracy and bias in the debased price can be explained almost entirely by the difference between the two RHSs, the return on portfolio assets for that **free to earn market return** on trading the market asset. By moving the expectation of the return on assets into the initial accurate premium returns a value that very nearly matches the debased premium

$$V_0^{debased} \approx V_0 - (\mu + \tfrac{1}{2}\sigma^2 - r) E\left(\int_0^T e^{-rt} S_t h_t dt\right)$$

whereby the debased price results from subsidizing the premium with the expectation of the economic value of the derivative. The remaining difference is the return earned on the larger delta positions for the replicating portfolio that replicates with deltas that contain return, those for the accurate premium, $h_t > h_t^{debased}$ (for a call). It is proved that the statistical bias is identically equal (within o(*dt*) variation) to this expectation for return on assets for the smaller debased deltas, i.e.,

$$\begin{aligned} V_0 - V_0^{debased} &= (\mu + \tfrac{1}{2}\sigma^2 - r) E\left(\int_0^T e^{-rt} S_t h_t^{debased} dt\right) \\ &= (\mu + \tfrac{1}{2}\sigma^2 - r)\int_0^T e^{-rt} E\left(S_t h_t^{debased}\right) dt \end{aligned}$$

This alternative formula for the statistical bias in the debased price is now proved. The logic exactly parallels that for the binomial walk whereby the slack variable Δ is removed from the probability distribution of the debased basis components. Unlike the walk, the slack variable is removed directly from the primed mean ($\mu' = \mu - \Delta$) rather than primed step probabilities ($p' = p - \Delta$ and $q' = q + \Delta$) or primed step ratios ($u' = u/\Delta$ and $d' = d/\Delta$). For an infinitesimal transition, the component slack variable in the recursion Δdt is shifted out of the debased basis components X' and into the expression for the premium by a change of variable and then

[8] If it had have worked [for zero NPV and flatlined options], then people would have realized sooner that doing so is completely pointless. Since it doesn't work, people have not realized; although they still would have had to ask themselves, *"what is the alternative?"* [and then been unfairly discredited]. This thesis has provided the alternative, with all of the supporting theory, and those who try to discredit it will only humiliate and discredit themselves.

separated out into a term $\Delta hSdt$ in proportion to ($\propto \Delta dt$) the path of delta h (and position in S) by a series expansion. This term, and its resulting path, is that debased from the PDE (partial derivative) and diffusion equation (total derivative) of the option premium with the constraint $d\Pi_{t,T}^{debased} = r\Pi_{t,T}^{debased} dt$. This is then repeated for each of the infinitesimal components of transition to option maturity and expiration of the deferral period of the derivative, over each possible path of the diffusion sample space, which results in a Riemann sum, and hence integral, of the infinitesimal components of slack separated out from the debased premium over the forward transition $T - t_0$ as a statistical expectation.

Explaining the Vanished Mean and the Bias is the Debased Premium

The premium debased by taking slack from μ: Sourcing the mispricing with trading returns earned by flatlining the option

For the statistically accurate premium and replicating portfolio that has not been debased, the backwards recursion for the premium is $V(k,T;S_t,t) = e^{-rdt}\int_0^\infty f_X(x)V(k,T;S_t \times x, t+dt)dx$. It was shown that unfolding and solving this recursion for the boundary conditions returns the statistically accurate premium V_0. Debasing the portfolio by killing the market return μ with slack $\Delta = \mu + \frac{1}{2}\sigma^2 - r$ to $\mu' = \mu - \Delta = r - \frac{1}{2}\sigma^2$, the recursion for the debased premium is then that for the accurate recursion but with the debased basis components X' that are primed with slack $V^{debased}(k,T;S_t,t) = e^{-rdt}\int_0^\infty f_{X'}(x)V^{debased}(k,T;S_t \times x, t+dt)dx$ which suppose that the forward transition of the asset for which the option is priced has no return net of cash so that the premium is void of market return. Solving $V^{debased}(k,T;S_t,t) = e^{-rdt}\int_0^\infty f_{X'}(x)V^{debased}(k,T;S_t \times x, t+dt)dx$ whilst keeping the basis components primed with slack for the boundary conditions returns the debased premium $V^{debased}$. Expressing the recursion for the debased premium with the accurate basis components by separating out the slack variable from the debased basis components by making a change of variable so that the slack variable is moved to the premium inside the deconvolution integral, and then separating out the slack variable in the expression for the premium with a series expansion so that the debased premium has time decay in the market asset [the asset for which is remeaned with no change in variance, unlike the binomial walk] rather than the demeaned asset, which then separates out the infinitesimal slack for one period of dt. This is then repeated for each period of dt, and each statistical possibility over each period resulting in a statistical expectation for the distribution of paths to option maturity. This then leaves a recursion solved with respect to the unprimed basis components and hence for the accurate premium V. The difference between the two premiums, V and $V^{debased}$, is then the statistical expectation over all possible paths of trading returns earned on the delta positions $h^{debased}$ in the market asset S; here we go…

The debased premium is $V^{debased}(k,T;S_t,t)$ and the recursion for the debased premium is

$$\begin{aligned}
V^{debased}(k,T;S_t,t) &= e^{-rdt}\int_0^\infty f_{X'}(x)V^{debased}(k,T;S_t \times x, t+dt)dx \\
&= e^{-rdt}\int_0^\infty \frac{1}{x\sigma\sqrt{2\pi\, dt}} e^{-\frac{1}{2}\frac{(\log x - \mu' dt)^2}{\sigma^2 dt}} V^{debased}(k,T;S_t \times x, t+dt)dx \\
&= e^{-rdt}\int_0^\infty \frac{1}{x\sigma\sqrt{2\pi\, dt}} e^{-\frac{1}{2}\frac{(\log x - \mu dt + \Delta dt)^2}{\sigma^2 dt}} V^{debased}(k,T;S_t \times x, t+dt)dx
\end{aligned}$$

where the basis density is primed with slack. Making a change of variable to extract the slack from the basis components (remove Δ from $f_{X'}$ so the recursion deconvolves for time decay with f_X free from Δ) and move it into the premium $V^{debased}$ in the deconvolution integral so it can be separated out in a series expansion of $V^{debased}$ for the term in Δ for the mispricing in $V^{debased}$

$$\begin{aligned}
\frac{\log x - \mu dt + \Delta dt}{\sigma} &= \frac{\log y - \mu dt}{\sigma} \\
\log x + \Delta dt &= \log y \\
y = xe^{\Delta dt} \qquad & x = 0 \Leftrightarrow y = 0 \\
dy = e^{\Delta dt}dx \qquad & x = \infty \Leftrightarrow y = \infty
\end{aligned}$$

$$V^{debased}(k,T;S_t,t) = e^{-rdt}\int_0^{\infty} \frac{1}{y\sigma\sqrt{2\pi\, dt}} e^{-\frac{1}{2}\frac{(\log y-\mu dt)^2}{\sigma^2 dt}} V^{debased}(k,T;S_t \times y e^{-\Delta dt}, t+dt)dy$$

where the basis density now deconvolves from *V* with respect to the market rate of return while the premium now contains the slack variable Δ, as a sink on the backward variable, which is now separated out in a power series. Expanding the option premium in a Taylor series for $f(e^{\log S_t \times y} e^{-\Delta dt})$ about $\log S_t y$

$$\begin{aligned}
&V^{debased}(k,T;S_t,t)\\
&= e^{-rdt}\int_0^{\infty} \frac{1}{y\sigma\sqrt{2\pi\, dt}} e^{-\frac{1}{2}\frac{(\log y-\mu dt)^2}{\sigma^2 dt}} \left\{ V^{debased}(k,T;S_t \times y, t+dt) - \Delta dt \frac{\partial}{\partial \log S_t y} V^{debased}(k,T;S_t \times y, t+dt) \right\} dy + o(dt)\\
&= e^{-rdt}\int_0^{\infty} \frac{1}{y\sigma\sqrt{2\pi\, dt}} e^{-\frac{1}{2}\frac{(\log y-\mu dt)^2}{\sigma^2 dt}} \left\{ V^{debased}(k,T;S_t \times y, t+dt) - \Delta dt \frac{\partial S_t y}{\partial \log S_t y} \frac{\partial}{\partial S_t y} V^{debased}(k,T;S_t \times y, t+dt) \right\} dy + o(dt)\\
&= e^{-rdt}\int_0^{\infty} \frac{1}{y\sigma\sqrt{2\pi\, dt}} e^{-\frac{1}{2}\frac{(\log y-\mu dt)^2}{\sigma^2 dt}} \left\{ V^{debased}(k,T;S_t \times y, t+dt) - \Delta dt S_t y \frac{\partial}{\partial S_t y} V^{debased}(k,T;S_t \times y, t+dt) \right\} dy + o(dt)\\
&= e^{-rdt}\int_0^{\infty} \frac{1}{y\sigma\sqrt{2\pi\, dt}} e^{-\frac{1}{2}\frac{(\log y-\mu dt)^2}{\sigma^2 dt}} V^{debased}(k,T;S_t \times y, t+dt)dy\\
&\qquad -\Delta dt\, e^{-rdt}\int_0^{\infty} \frac{1}{y\sigma\sqrt{2\pi\, dt}} e^{-\frac{1}{2}\frac{(\log y-\mu dt)^2}{\sigma^2 dt}} S_t y \frac{\partial}{\partial S_t y} V^{debased}(k,T;S_t \times y, t+dt)dy + o(dt)\\
&= e^{-rdt}\int_0^{\infty} \frac{1}{y\sigma\sqrt{2\pi\, dt}} e^{-\frac{1}{2}\frac{(\log y-\mu dt)^2}{\sigma^2 dt}} V^{debased}(k,T;S_t \times y, t+dt)dy\\
&\qquad -(\mu + \tfrac{1}{2}\sigma^2 - r)dt\, e^{-rdt}\int_0^{\infty} \frac{1}{y\sigma\sqrt{2\pi\, dt}} e^{-\frac{1}{2}\frac{(\log y-\mu dt)^2}{\sigma^2 dt}} S_t \times y \times h^{debased}(k,T;S_t \times y, t+dt)dy + o(dt)
\end{aligned}$$

where $h^{debased}(k,T;S_t,t)$ is the delta of the debased option where time decay in the option and the delta are with respect to the market asset *S* (with the basis density remeaned to *X* and hence with the correct parameter for drift) rather than the demeaned asset *Y* for which it is priced (the basis density primed with slack *X'*). This is for the first infinitesimal period *dt*. Similar principles apply for the second period whereby the slack variable in the recursion for the premium is similarly separated out of the primed basis components that then deconvolve for time decay in *S* rather than *Y*. Further unfolding

$$\begin{aligned}
V^{debased}(k,T;S_t,t) \;&= e^{-rdt}\int_0^{\infty} \frac{1}{y\sigma\sqrt{2\pi\, dt}} e^{-\frac{1}{2}\frac{(\log y-\mu dt)^2}{\sigma^2 dt}} V^{debased}(k,T;S_t \times y, t+dt)dy\\
&\qquad -(\mu + \tfrac{1}{2}\sigma^2 - r)dt\, e^{-rdt}\int_0^{\infty} \frac{1}{y\sigma\sqrt{2\pi\, dt}} e^{-\frac{1}{2}\frac{(\log y-\mu dt)^2}{\sigma^2 dt}} S_t \times y \times h^{debased}(k,T;S_t \times y, t+dt)dy + o(dt)
\end{aligned}$$

for the second infinitesimal

$$
\begin{aligned}
&V^{debased}(k,T;S_t,t)\\
&= e^{-rdt}\int_0^{\infty} \frac{1}{y_2\sigma\sqrt{2\pi\,dt}} e^{-\frac{1}{2}\frac{(\log y_2-\mu dt)^2}{\sigma^2 dt}} e^{-rdt}\int_0^{\infty} \frac{1}{y_1\sigma\sqrt{2\pi\,dt}} e^{-\frac{1}{2}\frac{(\log y_1-\mu dt)^2}{\sigma^2 dt}} V^{debased}(k,T;S_t\times y_1\times y_2,t+2dt)dy_1dy_2\\
&-\Delta dt\, e^{-rdt}\int_0^{\infty} \frac{1}{y_2\sigma\sqrt{2\pi\,dt}} e^{-\frac{1}{2}\frac{(\log y_2-\mu dt)^2}{\sigma^2 dt}} e^{-rdt}\int_0^{\infty} \frac{1}{y_1\sigma\sqrt{2\pi\,dt}} e^{-\frac{1}{2}\frac{(\log y_1-\mu dt)^2}{\sigma^2 dt}} S_t\times y_1\times y_2\times h^{debased}(k,T;S_t\times y_1\times y_2,t+2dt)dy_1dy_2\\
&\quad -\Delta dt\, e^{-rdt}\int_0^{\infty} \frac{1}{y_1\sigma\sqrt{2\pi\,dt}} e^{-\frac{1}{2}\frac{(\log y_1-\mu dt)^2}{\sigma^2 dt}} S_t\times y_1\times h^{debased}(k,T;S_t\times y_1,t+dt)dy_1+o(dt)\\
&= e^{-2rdt}\int_0^{\infty} \frac{1}{y\sigma\sqrt{2}\sqrt{2\pi\,dt}} e^{-\frac{1}{2}\frac{(\log y-2\mu dt)^2}{2\sigma^2 dt}} V^{debased}(k,T;S_t\times y,t+2dt)dy\\
&-(\mu+\tfrac{1}{2}\sigma^2-r)dt\, e^{-2rdt}\int_0^{\infty} \frac{1}{y\sigma\sqrt{2}\sqrt{2\pi\,dt}} e^{-\frac{1}{2}\frac{(\log y-2\mu dt)^2}{2\sigma^2 dt}} S_t\times y\times h^{debased}(k,T;S_t\times y,t+2dt)dy\\
&\quad -(\mu+\tfrac{1}{2}\sigma^2-r)dt\, e^{-rdt}\int_0^{\infty} \frac{1}{y\sigma\sqrt{2\pi\,dt}} e^{-\frac{1}{2}\frac{(\log y-\mu dt)^2}{\sigma^2 dt}} S_t\times y\times h^{debased}(k,T;S_t\times y,t+dt)dy+o(dt)
\end{aligned}
$$

which has one additional term for separated out slack with respect to twice the market return (2μ) and market risk (2σ, and hence statistical spread) for twice the length ($2dt$) of one infinitesimal forward transition of length dt , i.e., and integration over the forward transition distribution for a path of length $2dt$ and hence for the deconvolution of two dt basis density functions of X [rather than X']. Unfolding for the third infinitesimal

$$
\begin{aligned}
&V^{debased}(k,T;S_t,t)\\
&= e^{-3rdt}\int_0^{\infty} \frac{1}{y\sigma\sqrt{3}\sqrt{2\pi\,dt}} e^{-\frac{1}{2}\frac{(\log y-3\mu dt)^2}{3\sigma^2 dt}} V^{debased}(k,T;S_t\times y,t+3dt)dy\\
&-(\mu+\tfrac{1}{2}\sigma^2-r)dt\, e^{-3rdt}\int_0^{\infty} \frac{1}{y\sigma\sqrt{3}\sqrt{2\pi\,dt}} e^{-\frac{1}{2}\frac{(\log y-3\mu dt)^2}{3\sigma^2 dt}} S_t\times y\times h^{debased}(k,T;S_t\times y,t+3dt)dy\\
&\quad -(\mu+\tfrac{1}{2}\sigma^2-r)dt\, e^{-2rdt}\int_0^{\infty} \frac{1}{y\sigma\sqrt{2}\sqrt{2\pi\,dt}} e^{-\frac{1}{2}\frac{(\log y-2\mu dt)^2}{2\sigma^2 dt}} S_t\times y\times h^{debased}(k,T;S_t\times y,t+2dt)dy\\
&\quad\quad -(\mu+\tfrac{1}{2}\sigma^2-r)dt\, e^{-rdt}\int_0^{\infty} \frac{1}{y\sigma\sqrt{2\pi\,dt}} e^{-\frac{1}{2}\frac{(\log y-\mu dt)^2}{\sigma^2 dt}} S_t\times y\times h^{debased}(k,T;S_t\times y,t+dt)dy+o(dt)
\end{aligned}
$$

which again has one additional term for separated out slack, here with respect to thrice the market return (3μ) and market risk (3σ, and hence statistical spread) for thrice the length ($3dt$) of one infinitesimal forward transition of length dt , i.e., and integration over the forward transition distribution for a path of length $3dt$ and hence for the deconvolution of three dt basis density functions of X [rather than X']. The recursive pattern is now clear. Continuing for each of the infinitesimal steps to option maturity until the n infinitesimal steps to option maturity are unfolded

$$
\begin{aligned}
V^{debased}(k,T;S_t,t) &= e^{-nrdt}\int_0^{\infty} \frac{1}{y\sigma\sqrt{n}\sqrt{2\pi\,dt}} e^{-\frac{1}{2}\frac{(\log y-n\mu dt)^2}{n\sigma^2 dt}} V^{debased}(k,T;S_t\times y,t+ndt)dy_1dy_2\\
&\quad -(\mu+\tfrac{1}{2}\sigma^2-r)dt\sum\nolimits_{j=1}^{n} e^{-jrdt}\int_0^{\infty} \frac{1}{y\sigma\sqrt{j}\sqrt{2\pi\,dt}} e^{-\frac{1}{2}\frac{(\log y-j\mu dt)^2}{j\sigma^2 dt}} S_t\times y\times h^{debased}(k,T;S_t\times y,t+jdt)dy+o(dt)
\end{aligned}
$$

the integral is an expected value of the delta rebalances over every possible path of the diffusion sample space to option maturity and the summation is the sum over each individual path within the expectation so

$$
\begin{aligned}
V^{debased}(k,T;S_t,t) &= e^{-nrdt}\int_0^{\infty} \frac{1}{y\sigma\sqrt{n}\sqrt{2\pi\,dt}} e^{-\frac{1}{2}\frac{(\log y-n\mu dt)^2}{n\sigma^2 dt}} V^{debased}(k,T;S_t\times y,t+ndt)dy_1dy_2\\
&\quad -(\mu+\tfrac{1}{2}\sigma^2-r)\sum\nolimits_{j=1}^{n} e^{-jrdt}E\left(S_{t+jdt}h^{debased}(k,T;S_{t+jdt},t+jdt)\,|\,S_t\right)dt+o(dt)
\end{aligned}
$$

As $dt\to 0$, the path sums are Riemann integrals

$$V^{debased}(k,T;S_t,t) = e^{-nrdt}\int_0^{\infty} \frac{1}{y\sigma\sqrt{n}\sqrt{2\pi\, dt}} e^{-\frac{1}{2}\frac{(\log y - n\mu dt)^2}{n\sigma^2 dt}} V^{debased}(k,T;S_t \times y, t+ndt)dy_1 dy_2$$
$$-(\mu+\tfrac{1}{2}\sigma^2-r)\int_0^{T-t} e^{-rs}E\left(S_{t+s}h^{debased}(k,T;S_{t+s},t+s)\,|\,S_t\right)ds + o(dt)$$

The boundary condition for the debased option, at maturity, is the same as that for the statistically accurate premium. The premium on the right has been unfolded with the correct basis distribution, that with the slack separated out for time decay in the premium for every step of the forward transition to option expiry. This is that whereby the option depends, and the portfolio is rebalanced with, the market asset *S* rather than the demeaned asset for which the option is priced, **and which then expires for the [distribution of the] market asset *S* underlying the derivative** because of conditioning the backward variable $n=(T-t_0)/dt$ times for each possible filtration of length $n=(T-t_0)/dt$ infinitesimals dt. The filtration ends with a conditioning on the expiry value of the asset underlying the derivative. Hence

$$V^{debased}(k,T;S_t,t) = V(k,T;S_t,t) - (\mu+\tfrac{1}{2}\sigma^2-r)\int_0^{T-t} e^{-rs}E\left(S_{t+s}h^{debased}(k,T;S_{t+s},t+s)\,|\,S_t\right)ds + o(dt)$$

i.e.,

$$V_{t,T}^{debased} = V_{t,T} - (\mu+\tfrac{1}{2}\sigma^2-r)\int_t^{T} e^{-r(s-t)}E\left(S_s h_s^{debased}\,|\,S_t\right)ds + o(dt)$$

The statistical bias in the debased price is therefore the expected present value of the return on market assets of the replicating portfolio with the deltas of the debased option, i.e., the returns given up, earned on flatlining the option, in order to fund and subsidize the mispricing in the debased premium

$$Bias = V(k,T;S_t,t) - V^{debased}(k,T;S_t,t)$$
$$= (\mu+\tfrac{1}{2}\sigma^2-r)\int_t^{T} e^{-r(s-t)}E\left(S_s h_s^{debased}\,|\,S_t\right)ds + o(dt)$$

This is entirely analogues to the binomial situation with the difference, other than a continuous rather than binomial basis function and o(*dt*) variation, that updates must be made infinitely often and hence continuously.

By debasing a premium of this term, equivalent to killing μ with the slack variable Δ, debases the premium of the economic value of the derivative. Then expecting that this debased term be earned back, by earning trading returns with an offsetting position in the market asset that kills the payoff profile of the option, the option markets then cease to be markets for options but instead markets for redundant and worthless securities [that are subsidized by being flatlined] and a hellovalot of spot market trading. Besides the fact that this is **outright stupidity and a pointless circularity**, the inability to continuously trade and rebalance at ultra-high frequency, which means around the clock (24-7) right up to maturity to neutralize every single *dt* market fluctuation, invalidates any replicating argument in its entirety and any attempt at flatlining options. There is no correction for continuous time trading, nor corrections for market close with overnight periods or public holidays. The return-neutral argument under 'occasional rebalancing', for nothing other than a flatlined market, is pure fantasy.

■

Summary

The replicating arguments imply that

> **(1) i)** Someone who writes an underpriced call option at the debased price of Black and Scholes is subsidizing the purchaser of the call option (selling a call cheap) by subsidizing the mispricing in the debased price in advance by taking trading returns that will be earned on flatlining the option and moving them into the premium; in particular, the market returns earned from trading the underlying asset long in the replicating portfolio to flatline the call option are brought forward and used to reduce the price of the derivative;
>
> **(1) i)** Someone who writes an overpriced put option at the debased price of Black and Scholes is being subsidized by the purchaser of the put option (selling a put expensive) by using the mispricing in the debased price to finance in advance the trading losses that will be incurred on flatlining the option; in particular, the trading losses incurred from trading the underlying asset short in the replicating portfolio to flatline the put option are brought forward and funded by an increase in the price of the derivative;

Additionally, return-neutral valuation says that both sides of an option security must replicate and flatline their option positions, the purchaser (the long side) as well as the writer (the short side). So in terms of killing option positions for purchases

> **(2) i)** The cost saving for someone who purchases an underpriced call option at the debased price of Black and Scholes subsidizes the trading losses incurred on the short period-to-period holdings of the underlying market asset in the replicating portfolio used to flatline the option position;
>
> **(2) ii)** The additional cost for someone who purchases an overpriced put option at the debased price of Black and Scholes is subsidized by the trading returns earned on the long period-to-period holdings of the underlying market asset in the replicating portfolio used to flatline the option position.

This is a situation of subsidizing an option premium with trading returns earned on flatlining the option! As for the binomial walk, this is obviously a very pointless circularity unless you don't like options (because you think that they are redundant securities for example) in which case, the option shouldn't be written in the first place, and the option markets should be shut down. Then there will then be no option payoff profiles and no net cashflow, as the debased replicating arguments try to do but fail every time. If you want the option, then you must not replicate and flatline it with an offsetting asset. If the option is not flatlined with an offsetting replicating asset, then there is no means by which to earn back the mispricing, and the debased premium is outright wrong [as it is anyway].

The case where the portfolio returns are not held fixed and flat at r but are instead free to vary with **convex flux in market return**, the net cashflows of the option replication result in a net profit in the case of a call and a net loss in the case of a put, each a direct result of the investment earnings on trading long, or losses on trading short, the market asset underlying the derivative in order to kill the payoff profile of the deferred transaction. This profit, as earned revenue for making a market (or loss for taking the wrong side of the market, by taking up the pecuniary interest in a deferred short transaction for example), is also the theoretical result when people parameterize the premium and deltas with implied volatility and inadvertently [and inefficiently] get an undebased price consistent with the market by skewing the volatility in the debased price to [inadvertently] compensate for the statistical bias resulting from the slack variable in the mean, which is backed out into implied volatility from debased prices equated to market prices. Note that the term-to-maturity for the debased price equated to market prices must be in parity and must also correspond to the term-to-maturity for the option premium for which it parameterizes. Implied volatility depends on term-to-maturity; **(1)** monotonically decreasing with t_0 for a debased call for which implied vol is greater than market vol, and **(2)** monotonically increasing with t_0 for a debased put for which implied vol is smaller than market vol. This then invalidates the PDE of the debased premium [Black, Scholes, Merton], remeaned with skewing volatility with slack (to back out Δ), since this equation only applies to Brownian motion with constant volatility, which implied volatility is not because it depends on term-to-maturity. People who ignore the time dependence in implied volatility and use a flat implied volatility measure (say one month implied vol for a one month option) throughout the replication and option maturity period, ironically go on to overprice their calls (and deeper positions in the market asset) or underprice their puts, in which case, the profit on replicating and flatlining a call option or the loss on replicating and flatlining a put option is accentuated since there is an overcompensation for the statistical bias, i.e., more slack is introduced into the premium [via the volatility] than is contained in the premium by debasing it. This reverses the direction of the mispricing which gets worse (monotonically accentuates), for a fixed implied volatility measure, as the deferral period of the derivative is spent (decays) with increases in t_0 i.e., there is decay in the total compounded slack in the premium ($\Delta S h\, dt$ every infinitesimal) but the total amount of slack in the skewed volatility measure remains the same which then overcompensates as it is put back into the premium and deltas [for shortening terms to maturity] via volatility. There is not an exact formula for the degree of this overcompensation with term-to-maturity because [the proportional slack variable c of] skewed volatility backed out from market prices is numerical in the slack variable Δ (and k and $T - t_0$) but the consequences are obvious; more bias and less accuracy.

Depending on how much a debased price is skewed [and possibly remeaned] with implied volatility, statistical tests can be used to identify mispriced market options, from which mispriced options at debased prices can be exploited. A derivatives house that trades options at debased prices and does not skew their prices with implied volatility faces certain ruin when trading with those that exploit the mispricing and take the positive side of the bias for the derivative trades; buy calls and write puts, depending on how the prices are skewed with variations in volatility.

Similarly to gambling houses and casinos, who take the positive side of the statistical bias against the punters, this thesis is advocating working out the direction and magnitude of the statistical bias in the derivative markets [with statistical tests] which can then be exploited by those who are knowledgeable enough to know what statistical bias is and the effect that it has on statistically variable cashflow.

Statistical tests concern, among other techniques proposed by this thesis, **return-volatility ellipses**; **(1)** return-volatility confidence regions estimated (μ, σ) from spot market data and **(2)** implied return-volatility ellipses backed out from accurate premiums equated to market prices [rather than backing out implied volatility from debased premiums]. The ellipses then give the *x-y* axes; option prices can then be plotted as a surface on the *z* axis [to determine under and over priced options]. This then acknowledges and gives recognition to both risk and return, recognized in all other facets of finance, for decision making in derivatives. Other techniques include ANOVA and contrasts for various extracted basis functions.

Further, as also applicable to gambling houses and casinos, **ruin theory** in the context of the stochastic calculus can be utilized to estimate solvency probabilities which can be utilized as part of the risk management of derivative trading divisions. It is asserted that similar principles that are used to prudently manage and regulate insurance companies, in conjunction with the diffusion calculus of arbitrary order, be adopted for option trading divisions, market regulation, and promoting market efficiency. This requires, above all else, accuracy. Someone who cannot see that the variables of the calculus need to be accurate for such a mathematical management function is an abysmal and inept idiot and should be kept at arms distance from such a function [particularly if he is also a liar and fails PhDs for suggesting this (as I have suggested, and been failed, for a perfectly adequate and contributory thesis)]. Return-neutral valuation impairs all hope of being accurate and rational for the risk-management of derivatives; resulting in, for instance, inconsistent numbers for decision making and hence failure of management systems which depend on these numbers for decision criteria. Risk-neutral is systematically in error, and practitioners of this flawed idiocy cannot hope to get anything right, and have no hope of preventing disasters, as consistently observed in derivative markets around the world in recent times.

Explaining the Statistical Bias in the Debased Premium out of the Context of a Replicating Portfolio

If you want the payoff profile of your option position, you must not take an offsetting position in a replicating asset since this [attempts to] flatlines the payoff profile of the option [but fails always]. In such a circumstance, an option at the debased price is a redundant and worthless security which should not have been traded in the first place. Out of the context of taking an offsetting position in the market asset, which kills the option delta and the payoff profile of the option to earn back the statistical mispricing, the debased price is outright wrong since there is no means to earn back the mispricing in the debased premium. In any circumstance, with a replication or not, the debased premium is outright wrong for the pricing of options.

If options are not to be redundant and worthless securities, the economic value of derivatives must be acknowledged by recognizing the return of the underlying spot markets for which the derivative markets depend as deferred transactions of these markets. The moral of this story is that premiums for options need to be priced accurately, statistically and hence without statistical bias [by killing μ with slack Δ], for the market assets on which they depend [if this is not already half obvious]. This will then bring theory of derivatives into line with other facets of finance for a consistency with technical analysis, as well as CAPM and Efficient Frontier.

RETURNING DEBASED PRICES TO ACCURATE PRICES

The debased option prices are statistically biased and inaccurate, and are priced with no market return because of the slack variable Δ that debases it all. These are the wrong prices for options on market variables, and are instead priced for the oxymoron bank account Y that is notionally void of return yet has full market risk. For market assets that earn more than cash, the debased call option premium underprices call options and the debased put option premium overprices put options. Considering the statistically accurate and unbiased premiums for options on the market asset S

$$V(k,T;S_t,t) = S_t e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t)}\overline{F}_U\left(u_k-\sigma\sqrt{T-t}\right)-ke^{-r(T-t)}\overline{F}_U\left(u_k\right) \qquad \text{for a call option}$$

$$V(k,T;S_t,t) = ke^{-r(T-t)}F_U\left(u_k\right)-S_t e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t)}F_U\left(u_k-\sigma\sqrt{T-t}\right) \qquad \text{for a put option}$$

where U is a standard unit normal $U \sim N(0,1)$, F_U and $\overline{F}_U$ are the cumulative and complementary cumulative distribution functions of the standard normal U, S_t is the spot price of the market asset underlying derivative, and u_k is the log-standardized value of the exercise price with respect to S and a normal pivot for $U \sim N(0,1)$

$$u_k = \frac{\log k - \log S_t - \mu(T-t)}{\sigma\sqrt{T-t}}$$

If one takes the expected value of the asset to deliver at option expiration $E\big(S(T)\,|\,S(t)\big)$

$$E\big(S(T)\,|\,S(t)\big) = S_t e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t)}$$

and lets this be $Y_t = S_t e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t)}$ by conditioning on the backward variable of the demeaned asset *Y*, then the future value of the forward transition variable of the demeaned asset *Y* consists of a mean at the cash rate within the demeaned transition distribution but starting from this greater value. The forward transition of $Y(T)\,|\,Y(t) = S_t e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t)}$ then expires with the same transition distribution as $S(T)\,|\,S(t) = S_t$. The transition distribution of the demeaned asset *Y* then makes it to the asset to deliver at option expiry and is hence accurate. The equations of the accurate premiums, by writing Y_t for $S_t e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t)}$, then become

$$V(k,T;Y_t,t) = Y_t \overline{F}_U\left(u_k - \sigma\sqrt{T-t}\right) - ke^{-r(T-t)}\overline{F}_U\left(u_k\right) \qquad \text{for a call option}$$

$$V(k,T;Y_t,t) = ke^{-r(T-t)}F_U\left(u_k\right) - Y_t F_U\left(u_k - \sigma\sqrt{T-t}\right) \qquad \text{for a put option}$$

for $u_k = \dfrac{\log k - \log Y_t e^{-(\mu+\frac{1}{2}\sigma^2-r)} - \mu(T-t)}{\sigma\sqrt{T-t}} = \dfrac{\log k - \log Y_t - (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$ (which is the log-standardized exercise price with respect to *Y*) and $U \sim N(0,1)$.

These put and call formulae are symbolically the same as the debased formulae except that Y_t is not the spot price, but rather the spot adjusted by the future value of the returns of the market asset *S* over the deferral period of the derivative. Upon evaluation, these are the unbiased premiums as conditioned on the backward variable $Y_t = S_t e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t)}$ in which case the forward transition distributions of *S* and *Y* coincide, i.e.,

$$\begin{aligned} Y(T)\,|\,Y(t) &= S_t e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t)} \\ &\sim \log N\left(\log S_t e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t)} + (r-\tfrac{1}{2}\sigma^2)(T-t),\, \sigma^2(T-t)\right) \\ &\sim \log N\left(\log S_t + \mu(T-t),\, \sigma^2(T-t)\right) \end{aligned}$$

otherwise, the transition distributions do not correspond, i.e., whenever $Y_t \neq S_t e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t)}$ and the forward variable of the demeaned asset has not been re-meaned by assigning it a source to account for the return of the underlying spot market over the deferral period of the derivative.

Black and Scholes use the variable Y_t as the spot price of the market asset $Y_t = S_t$. This gives the wrong transition distribution

$$\begin{aligned} Y(T)\,|\,Y(t) &= S_t \\ &\sim \log N\left(\log S_t + (r-\tfrac{1}{2}\sigma^2)(T-t),\, \sigma^2(T-t)\right) \end{aligned}$$

and the wrong premium, the difference between the resulting premiums being that of the statistical bias and inaccuracy of the debased price. This corresponds one-for-one with taking the distribution of the market asset, killing the market return of the market asset with the slack variable [surreptitiously by holding $d\Pi_t = r\Pi_t dt$] and then using this to price the option, i.e., that for *Y* conditional on $Y(t) = S_t$ as the spot price.

Converting the Debased Option Premium and PDE to that Accurate for S

Remeaning the backward variable of *Y* then gives a method of remeaning the debased price to get an accurate price. **(1)** Firstly, acknowledge that the debased premium is the premium priced for *Y* and write out the formulae for the debased option premium with Y_t as the spot price for the notional demeaned asset. **(2)** Then, to acknowledge the market return of the underlying market asset, take the spot price S_t of the underlying market asset *S* and accumulate it by the future value of the underlying spot market $e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t)}$ over the deferral period of the derivative. Set this equal to the spot price of the demeaned asset $Y_t = S_t e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t)}$, which otherwise ignores the economic value of the underlying spot market over the deferral period of the derivative. **(3)** Then

substitute this into the debased formula for *Y*. What you get is the premium accurate and unbiased for the underlying market asset *S*

$$V(T,k;Y_t,t) = Y_t \overline{F}_U\left(u_k - \sigma\sqrt{T-t}\right) - ke^{-r(T-t)}\overline{F}_U\left(u_k\right)$$
$$= S_t e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t)}\overline{F}_U\left(u_k - \sigma\sqrt{T-t}\right) - ke^{-r(T-t)}\overline{F}_U\left(u_k\right) \quad \text{for a call option}$$

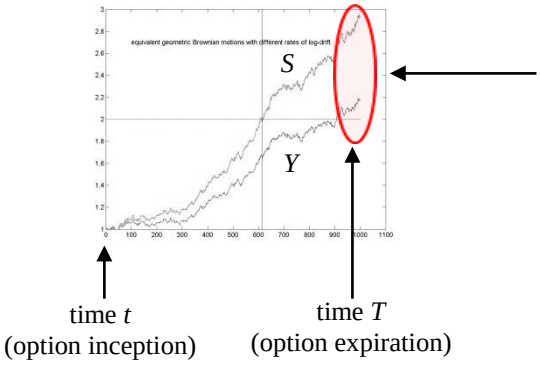
$$V(T,k;Y_t,t) = ke^{-r(T-t)}F_U\left(u_k\right) - Y_t F_U\left(u_k - \sigma\sqrt{T-t}\right)$$
$$= ke^{-r(T-t)}F_U\left(u_k\right) - S_t e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t)}F_U\left(u_k - \sigma\sqrt{T-t}\right) \quad \text{for a put option}$$

$$\text{for } u_k = \frac{\log k - \log S_t e^{(\mu+\frac{1}{2}\sigma^2-r)} - (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = \frac{\log k - \log S_t - \mu(T-t)}{\sigma\sqrt{T-t}}$$

(4) QED. Write it out for yourself and see. This is equivalent to transforming the backward variable of the demeaned asset *Y*, within the debased premium, to that of the market asset *S*. This method directly removes the slack variable Δ that debases return and therefore returns the debased price to the accurate price. This method also remeans the backward parabolic PDE of the debased price by returning the debased component of flux $\Delta S_t h_t$ to the PDE [which was surreptitiously debased by holding the portfolio returns fixed flat at cash with the artificial constraint $d\Pi = r\Pi dt$]. This results in the backward parabolic PDE of the statistically accurate premium [and portfolio free to earn return and the economic value of the derivative]. Further, this is a better method to get an accurate price than skewing volatility which introduces another [proportional] slack variable *c* to compensate for the [additive] slack variable Δ introduced into the mean to debase returns. In contrast to removing this variable Δ directly, which is therefore minimum-variance, skewing volatility is not minimum-variance because of introducing further variance with *c* in addition to that in Δ [rather than simply removing Δ], and is therefore sub-optimal. It also introduces a non-constant parameter into the dynamics [for a replicating portfolio] since the slack *c* in implied volatility is dependent on, and hence varies with, term-to-maturity. So instead of implying volatility from market prices, which has skew in *c* generated by Δ [with respect to *k*], this thesis suggests and argues that returns [rather than volatility] should be implied from option market prices. Returns are implied from option market prices by reference to **market put-call parity**; i.e., $S_t e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t)}$ implied by the market value of $c_{t,T} - p_{t,T} + ke^{-r(T-t)}$, which is set equal to Y_t in the debased premium to obtained an accurate premium. This is consistent with technical views of the underlying spot markets, and then corrects for all three indicators that debased prices are not consistent with market prices, and instead that the option markets price returns of the spot market into premiums for options on spot market variables

1) **market prices** are not debased (and have never equated to values obtained by the debased formulae)
2) **debased put-call parity** does not hold for market option prices
3) **implied volatility** varies systematically with exercise price

where **(2)** and **(3)** are simply evidence of **(1)**. Remeaning the backward variable accounts for these facts simply by removing the slack variable Δ, the variable which debases derivatives and consequently generates all of these observations which are nothing other than evidence that debased prices are wrong prices.



the difference between the two boundary conditions and the two assets (that of *S* and that of *Y*) for a conditioned backward variable at the spot price $S_t = Y_t$;

In this diagram, the demeaned asset at the option maturity only makes it to the asset to deliver *S*(*T*) if the lower path *Y* is moved horizontally upwards at the backward variable at option inception at time *t*, from $Y_t = S_t$ to $Y_t = S_t \exp\left((\mu + \frac{1}{2}\sigma^2 - r)(T-t)\right)$. This transformation and resulting upwards parallel shift of the path *Y* remeans the deameaned asset and corrects the return-free option premium. Alternatively, the gradient can be increased by remeaning the forward variable.

The point here is how the backward variable of the demeaned asset *Y* is conditioned in relation to that of the market asset *S*. By conditioning, the backward variable of the demeaned asset can be given any value but only when it is equal to the accumulated backward variable (accumulated by the market return) does the formula give

the correct premium corresponding to the correct forward transition of the market asset *S*. Consequently, if you take the accurate premium and then demean the spot price of its future value, one arrives at the Black and Scholes premium for the debased price of an option derivative. This is exactly equivalent to debasing the accurate premium with the slack variable Δ [surreptitiously by holding portfolio returns fixed flat at cash]. Hence, calls are underpriced and puts are overpriced for market assets that earn more than nothing, net of cash, i.e., for any market where long positions are more profitable than short positions. Similarly, if one takes the $N(d_2)$ term in the debased call option formula and calculates this probability at the adjusted spot price to account for the spot markets future value, $Y_t = S_t \exp\left((\mu + \frac{1}{2}\sigma^2 - r)(T-t)\right)$, then it is a correct probability since it remeans the forward transition by removing the slack that junks the probability with bias. If it is calculated taking $Y_t = S_t$ as the spot price, then the forward transition remains junked with slack and statistical bias and hence is a wrong probability. Hence, as an alternative to solving the PDE of the statistically unbiased and accurate option $-V_t = \frac{1}{2}\sigma^2 S^2 V_{SS} + (\mu + \frac{1}{2}\sigma^2) S V_S - rV$ or calculating the formulae for the unbiased and accurate premium directly, the unbiased values can be obtained by means of either the PDE of the demeaned asset or directly from the premium of the demeaned asset. This regards either a vertical shift upwards of the [logarithm of the] demeaned path, corresponding to a transformation of the backward variable *Y* (and consistent with giving the spot price a future value as suggested by the market), or increasing the slope of the demeaned path which corresponds to transforming the forward variable *Y* to remean the transition distribution for the correct drift [gradient] of the market asset to deliver *S*. Within the context of a PDE, **(1)** remeaning the backward variable is equivalent to transforming the independent variable [for the diffusion derivatives] and **(2)** transforming the forward variable is equivalent to transforming the boundary condition. In other words, solving the debased PDE or calculating the debased premium with respect to either

$$\textbf{(1)}\ \max\left(Y(T)-k,0\ \middle|\ Y(t)=S_t e^{(\mu-\frac{1}{2}\sigma^2-r)(T-t)}\right) \qquad \text{or} \qquad \textbf{(2)}\ \max\left(Y(T)e^{(\mu-\frac{1}{2}\sigma^2-r)(T-t)}-k,0\ \middle|\ Y(t)=S_t\right)$$

remeans the debased price **(1)** by transforming the backward variable $Y(t)$ to give it a future value by vertically shifting the path of *Y* upwards or **(2)** by assigning a source to the forward variable to remean and make $Y(T)$ bigger to correspond to $S(T)$ by increasing the gradient of the path of *Y*. Either of these converts the debased PDE and premium to the accurate PDE solved for, and premium calculated with respect to,

$$\max\left(S(T)-k,0\ \middle|\ S(t)=S_t\right)$$

as per the correct and accurate forward transition distribution of the market asset *S* underlying the derivative. This method which assigns a source to the forward transition [via either the forward variable or the backward variable] and transforms the Black-Scholes-Merton PDE, which is wrong for the asset to deliver, to the correct PDE $rV - V_t = \frac{1}{2}\sigma^2 S^2 V_{SS} + (\mu + \frac{1}{2}\sigma^2) S V_S$ that is accurate for the asset to deliver [derived by and original to this thesis] by removing the slack variable that debases the flux of ΔSh. Transforming the forward variable or the backward variable are equivalent due to time homogeneity of first order diffusion. By applying a transformation to the premium [rather than boundary or initial condition of the debased PDE], remeaning the premium by reference to boosting the spot price corresponds to transforming the backward variable. If it is the forward variable that is transformed, then to get the accurate premium from the debased premium, replace every instance of r with $\mu + \frac{1}{2}\sigma^2$ and then adjusts the premium discounting by multiplying the result by $\exp\left((\mu + \frac{1}{2}\sigma^2 - r)(T-t)\right)$; this is not as mathematically intuitive as adjusting the spot price by reference to the put-call parity relationship [and is hence not the method preferred by this thesis]. Aside from these two methods, there are two other suboptimal methods to remean a debased price. A sink can be assigned to the option exercise price *k* and the premium accumulated at the excess return

$$e^{(\mu-\frac{1}{2}\sigma^2-r)(T-t)} \max\left(Y(T)-ke^{-(\mu-\frac{1}{2}\sigma^2-r)(T-t)},0\ \middle|\ Y(t)=S_t\right)$$

This is a rearrangement of transforming the forward variable of the demeaned asset to that of the market asset to deliver by remeaning it. It is essentially removing the slack variable which debases return over the systemic variation observed in implied volatility as a function of the exercise price *k*, which is the point [*k*] for which the slack variable is then being reintroduced to cancel its effects on return, i.e., via the variable *k* which introduces the asymmetry into options. The other alternative is to skew volatility, by compensating for the slack variable Δ in return with another slack variable *c* in volatility.

There are hence four methods of returning a debased price to an accurate price, i.e., four methods of removing [or compensating for] the slack variable Δ that kills the market return in the accurate premium. These are

(1) transforming the backward diffusion variable [transforming independent variable to remove Δ]
(2) transforming the forward diffusion variable [transforming boundary condition to remove Δ]

(3) transforming the exercise price
(4) skewing volatility [skewing the variance with slack $c(\Delta)$ to compensate for Δ]

Each of these methods, other than skewing volatility, are transformations of the premium and PDE of the demeaned asset Y and result exactly in the premium and PDE that are accurate for the market asset to deliver S. Note that some return-neutral practitioners call skewing volatility to get a more sensible premium than the debased premium "tweaking" the Black Scholes price [i.e., the debased price]. Ironically, by doing this, they are acknowledging that the debased price is wrong, since they are inadvertently just trying to remove the slack variable [introduced with $d\Pi_t = r\Pi_t dt$] that debases the drift. Return-neutral valuation skewered itself [and this thesis delivers the mate]. Although useful for getting a feel for the relationships between the option price and the option parameters, skewing up volatility to increase the debased price of a call, or skewing down volatility to decrease the debased price of a put, is suboptimal for getting an accurate price consistent with the market. Of the four methods to remean a debased price, skewing volatility is the method that is least consistent with **minimum variance unbiased estimation**, in this case, of option prices. Skewing volatility is considered now.

THE VOLATILITY SKEW

Introduction

Lemma: Δ generates the skew in volatility

It is commonly known and widely discussed in the [credible] literature that **(1)** option prices are not Black and Scholes, **(2)** that put call parity does not hold for return-free options [debased put-call parity] and **(3)** volatility when backed out of the Black and Scholes formulae equated to market prices is a systematic function of the exercise price. These undeniable observations are clear evidence that option market prices are not debased, and instead, tend towards unbiased estimates for option prices that explicitly recognize the return of the market assets $S(T)|S(t)$ underlying the derivatives as deferred transactions of the spot markets. Assuming that, through interaction between supply and demand and the instinctive feel that large numbers of traders and market makers get for the risks that they trade, the market makes sensible best estimates of the risks for which options are written and prices them accordingly by providing a view on the unbiased premia that accurately reflects its assessment of the risk and return of the spot markets. Then, by equating the debased prices with market prices [that do not kill μ with slack Δ], volatility as implied from the debased price must take up the slack[9] for the mis-specified rate of return within the premium of the debased price, i.e., the volatility takes up the slack within variance for the slack introduced into the mean after it is debased with Δ. This slack, for volatility, is then dependent upon

(1) whether the option is a put or a call
(2) the level of the exercise price, k
(3) the term to expiry, $T - t_0$
(4) the divergence between return and no return net of cash, i.e., the size of the slack variable Δ that generates the bias and consequential skew in volatility

and is accounted for by skewing the volatility to make up the divergence between return and no return and hence the level of the debasing slack, i.e., adjusting the second cumulant to compensate for the first. The volatility skew is then obtained by solving the equation resulting from the equated prices, the markets assessment of the statistically accurate and unbiased premium (as per market prices explicitly recognizing the return of the underlying spot market for which the transaction is being deferred) and the debased price with skewed volatility. Setting the volatility skew factor c as proportional to the volatility parameter

$$\sigma^{IV} = c\sigma \qquad \text{where } c = c(k, T - t_0, \Delta)$$

where the skew factor c is another slack variable which backs out the slack variable Δ introduced via returns into volatility, as a function of the exercise price and term-to-maturity.

The resulting equation obtained by equating the market price [which acknowledges market return] to the debased premium [which does not] then generates the volatility skew in k [from the asymmetry of options] and hence explains the skew of implied volatility. The solution must be solved numerically since the proportional slack variable c is not an analytic function. The example following the theory does this with Mathematica. An

[9] as per the definition of a slack variable in linear programming and operations research; in this case, the slack variable c is taken as proportional (rather than additive in O.R. or the debasing slack in the drift) so that $h(\sigma) - g(c\sigma) = 0$ where c is the slack variable, h is the unbiased premium, and g is the debased premium.

analytical formula that closely approximates the solution for this slack, and the consequent volatility skew as a function of the exercise price, is obtained for outofthemoney options.

Preliminaries

Calls are underpriced and puts are overpriced by debased premiums and both calls and puts are positive functions of volatility. Then from null skew at market volatility, $c = 1$,whenever an option is underpriced volatility needs to be nudged upwards ($c \uparrow$ and consequently $c > 1$) and whenever an option is overpriced volatility needs to be nudged downwards ($c \downarrow$ and consequently $c < 1$).

Skewing volatility in this way $\sigma^{IV} = c\sigma$ ($c = c(k)$) to account for the statistical bias and statistical inaccuracy of the debased formulae then ***explains the volatility skew****.*

Underpriced call options mean overpriced put options. The debased premium underprices call options and overprices put options whenever the market asset earns more than a bank account [which is a standard assumption in all other financial theory other than return-neutral valuation]. To compensate for the statistical bias and take up the slack for a fixed exercise value, then, volatility of a call option must be skewed upwards ($c > 1$) and volatility of a put option must be skewed downwards ($c < 1$). Moreover, a call option premium is a decreasing function of the exercise price and a put option premium is an increasing function of the exercise price. A debased call option premium must be increased to compensate for the statistical bias by increasing volatility and a debased put option premium must be decreased to compensate for the bias by decreasing volatility; therefore, the volatility must be compensated in the same direction for both puts and calls as they vary as functions of the exercise price k. For a call (for a right to buy @ k), the bigger the exercise value k, the less inthemoney the option is and the less material the statistical bias and so the less volatility must compensate upwards when it is backed out of the demeaned option premium. For a put (for a right to sell @ k), the bigger the exercise value, the more inthemoney the option is and the more material the statistical bias and so the more volatility must be compensated downwards when it is backed out. Therefore skewed volatility is a monotonic decreasing function of the exercise price k [over the well-defined regions of the solution for k; heavily in the money options do not have sensible solutions for the slack variable and the consequent volatility skew, i.e., implied volatility is not defined]. For the regions that skewed volatility is defined, the slack variable c of the skew as a function of the exercise price k is greater than one for calls ($c > 1$, increasing the price) and less than one for puts ($c < 1$, reducing the price); moreover, for increases in k, the slack variable (and hence skew) c approaches one from above for calls and diverges downwards away from one for puts. For put options, as soon as the slack variable c has a root, and changes sign, implied volatility is not defined.

The Mathematics of the Volatility Skew: the volatility slack $c(\Delta)$ between return and no return

Letting the nudge factor c to skew the volatility be proportional to σ, and let this be proportional slack. The volatility skew, for the implied volatility in terms of the proportional slack variable, is then $c\sigma$. Say the option market acknowledges the return of the underlying spot markets and makes a sensible best estimate of the statistically accurate and unbiased premium for the market asset to deliver S by acknowledging the return of the underlying spot markets with forward transition distribution

$$S(T)\,|\,S(t) = S_t$$
$$\sim \log N\left(\log S_t + \mu(T-t),\ \sigma^2(T-t)\right)$$

for which the premiums are either

$$V(k,T;S_t,t) = e^{-r(T-t)}\int_k^{\infty} (x-k)\, f_{S(T)|S(t)}(x\,|\,S_t)dx \qquad \text{for a call option}$$

$$V(k,T;S_t,t) = e^{-r(T-t)}\int_0^{k} (k-x)\, f_{S(T)|S(t)}(x\,|\,S_t)dx \qquad \text{for a put option}$$

where S_t is the spot price of the market asset. The debased premiums are those for the demeaned [return-free] asset Y [that of S with μ killed with Δ]

$$Y(T)\,|\,Y(t) = S_t$$
$$\sim \log N\left(\log S_t + (r-\tfrac{1}{2}\sigma^2)(T-t),\ \sigma^2(T-t)\right)$$

for which the premiums are either

$$V^{debased}(k,T;S_t,t) = e^{-r(T-t)}\int_k^{\infty}(x-k)\,f_{Y(T)|Y(t)}(x\,|\,S_t)dx \qquad \text{for a call option}$$

$$V^{debased}(k,T;S_t,t) = e^{-r(T-t)}\int_0^{k}(k-x)\,f_{Y(T)|Y(t)}(x\,|\,S_t)dx \qquad \text{for a put option}$$

with Y_t set to the spot price of the market asset $Y_t = S_t$ as per the debased price for an option derivative. Volatility skew $c(k)$ is then solution to the equation

$$e^{-r(T-t)}\int_k^{\infty}(x-k)\,f_{S(T)|S(t)}(x\,|\,S_t)dx = e^{-r(T-t)}\int_k^{\infty}(x-k)\,f_{Y(T)|Y(t)}(x\,|\,S_t)dx \qquad \text{for call option implied vol.}$$

or the equation

$$e^{-r(T-t)}\int_0^{k}(k-x)\,f_{S(T)|S(t)}(x\,|\,S_t)dx = e^{-r(T-t)}\int_0^{k}(k-x)\,f_{Y(T)|Y(t)}(x\,|\,S_t)dx \qquad \text{for put option implied vol.}$$

where σ in the functions on the right hand side for the expectation in the forward transition for Y are replaced with $\sigma^{IV} = c(k)\sigma$, i.e., a premium based on the demeaned [return-neutral] transition distribution of the demeaned asset Y, taking the backward variable as the spot price $Y_t = S_t$, but with skewed volatility

$$\begin{aligned} &Y(T)\,|\,Y(t) = S_t \\ &\quad \sim \log N\left(\log Y(t) + (r - \tfrac{1}{2}c^2\sigma^2)(T-t),\ c^2\sigma^2(T-t)\right) \\ &\quad \sim \log N\left(\log S_t + (r - \tfrac{1}{2}c^2\sigma^2)(T-t),\ c^2\sigma^2(T-t)\right) \end{aligned}$$

The equation for which needs to be solved numerically for c by letting k vary for the slack $c = c(k)$. This is easily done with a root finder. The equation

$$\begin{aligned} &S_t e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t)}\overline{F}_U\left(\frac{\log k - \log S_t - \mu(T-t)}{\sigma\sqrt{T-t}} - \sigma\sqrt{T-t}\right) - ke^{-r(T-t)}\overline{F}_U\left(\frac{\log k - \log S_t - \mu(T-t)}{\sigma\sqrt{T-t}}\right) \\ &= S_t\overline{F}_U\left(\frac{\log k - \log S_t - (r - \frac{1}{2}c^2\sigma^2)(T-t)}{c\sigma\sqrt{T-t}} - \sigma\sqrt{T-t}\right) - ke^{-r(T-t)}\overline{F}_U\left(\frac{\log k - \log S_t - (r - \frac{1}{2}c^2\sigma^2)(T-t)}{c\sigma\sqrt{T-t}}\right) \end{aligned}$$

solves for c for the slack variable in the call option implied volatility, and the equation

$$\begin{aligned} &ke^{-r(T-t)}F_U\left(\frac{\log k - \log S_t - \mu(T-t)}{\sigma\sqrt{T-t}}\right) - S_t e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t)}F_U\left(\frac{\log k - \log S_t - \mu(T-t)}{\sigma\sqrt{T-t}} - \sigma\sqrt{T-t}\right) \\ &= ke^{-r(T-t)}F_U\left(\frac{\log k - \log S_t - (r - \frac{1}{2}c^2\sigma^2)(T-t)}{c\sigma\sqrt{T-t}}\right) - S_t F_U\left(\frac{\log k - \log S_t - (r - \frac{1}{2}c^2\sigma^2)(T-t)}{c\sigma\sqrt{T-t}} - \sigma\sqrt{T-t}\right) \end{aligned}$$

solves for c for the slack variable in put option implied volatility. The left hand sides for the accurate put and call option premiums are given by the market which recognizes the return of the underlying spot market. For the parameters on the right hand side, the cash rate is observed from Government bonds, the spot price S_t is observed from the spot market, and the exercise price and term-to-maturity are the particulars of the option agreement. The slack $c(\Delta)$ for implied volatility is backed out from that implied by the market, from the market price equated to the debased price, and is a systematic skew as a function of the exercise price k. A numerical example is provided with Mathematica, under the assumption that the market makes an accurate estimate of the accurate premium that recognizes market return μ [does not kill it with Δ] of the underlying spot market.

Volatility Skew Approximation

As a simple approximation, making the expiry probabilities correct and ignoring the volatility slack in $\frac{1}{2}\sigma^2$ of the drift of the debased price, then

$$\overline{F}_{N(0,1)}\left(\frac{\log k - \log S_t - \mu(T-t)}{\sigma\sqrt{T-t}}\right) = \overline{F}_{N(0,1)}\left(\frac{\log k - \log Y_t - (r - \frac{1}{2}\sigma^2)(T-t)}{c\sigma\sqrt{T-t}}\right)$$

that is

$$\frac{\log k - \log S_t - \mu(T-t)}{\sigma\sqrt{T-t}} = \frac{\log k - \log Y_t - (r - \frac{1}{2}\sigma^2)(T-t)}{c\sigma\sqrt{T-t}}$$

which solves for

$$c = 1 + \frac{(\mu + \frac{1}{2}\sigma^2 - r)(T-t)}{\log k - \log S_t - \mu(T-t)} = 1 + \frac{\Delta(T-t)}{\log k - \log S_t - \mu(T-t)}$$

The implied volatility is then approximately

$$\sigma^{\text{Implied}} = c\sigma = \sigma + \frac{\Delta\sigma(T-t)}{\log k - \log S_t - \mu(T-t)} \qquad \text{for} \qquad \begin{cases} k > S_t e^{\mu(T-t)} & \text{for a call} \\ k < S_t e^{\mu(T-t)} & \text{for a put} \end{cases}$$

which gives a simple guiding formula for the characteristics of $c = c(k, \Delta, T-t)$. See for yourself with the Mathematica graphs. This approximation for the volatility skew is defined over the region $k > S_t e^{\mu(T-t)}$ for outofthemoney call options and $k < S_t e^{\mu(T-t)}$ for outofthemoney put options. Similarly to the approximation, the exact skew is not defined for moderately to very inthemoney options. This volatility function increases ($c = 1 + \delta$) with the exercise price in proportion to the excess return [and slack variable] $\mu + \frac{1}{2}\sigma^2 - r$ and term-to-maturity, and the inverse of the logarithm of the exercise price *k*. This function is a monotonic decreasing function of *k* and hence a smirk with respect to *k*. This is, roughly speaking, an inverse log hyperbolae as is, roughly, the exact numerical skew. The analytic approximation gives a good guide to the behavior of the exact numerical solution where it is sensibly defined. There is no volatility skew [no systematic arc] for debased option premiums when, and only when, market assets have no return in which case the debased option premium is then statistically unbiased and hence **free from the slack Δ that generates the skew**. In the approximation, $\sigma^{\text{Implied}} = \sigma$ when $\Delta = 0$, i.e., implied volatility σ^{Implied} is in parity with market volatility σ [a horizontal line with respect to *k*] when the debasing slack variable Δ is zero, as it should be.

Note that people that advocated implied volatility, rather than market volatility, for Black and Scholes prices knew [with instinct rather than maths] that the debased price was the wrong price and were demonstrating so by advocating this. People use implied volatility rather than market volatility for debased prices because the people who advocated this were right in advocating so. The debased price is the wrong price. The debased price needs to be remeaned to the accurate price [that recognizes μ] if it is to be a sensible price.

Numerical Example in Mathematica: Call Option

- Defining the function for the Call Option Premium (unbiased and accurate) in Mathematica

```
V[k_, S_, m_, s_, r_, t_] :=
 S Exp[(m - 0.5 s^2 - r) t]
   (1 - CDF[NormalDistribution[0, 1], (Log[k / S] - m t) / Sqrt[s^2 t]]) -
  k Exp[-r t] (1 - CDF[NormalDistribution[0, 1], (Log[k / S] - m t) / Sqrt[s^2 t]]);
```

- check that it works; Say the strike is $10, the spot is $12, the mean return on the stock is 15% pa, the volatility is 15% pa, the cash rate is 5% pa, and the term-to-maturity is 1 year;

unbiased premium: `V[10, 12, .15, .15, .05, 1]` = $3.90616

debased premium: `V[10, 12, .03875, .15, .05, 1]` = $2.5296

bias: `CallBias := V[10, 12, .03875, .15, .05, 1] - V[10, 12, .15, .15, .05, 1]` = - $1.37656

Implied vol to get accurate premium: `FindRoot[V[10, 12, .05 - 0.5 s^2, s, .05, 1], {s, .151}]` = 0.58319 pa

Implied volatility is 58.319% pa, which is greater the market volatility of 15% pa. This is because the debased price underestimates the accurate price (by $1.3766) and therefore volatility must be skewed upwards to increase the debased price to bring it into parity. In this case, it is more than three times (*c* = 3.89) that of the original volatility.

Debased premium with implied vol: `V[10, 12, .05 - 0.5 0.58532^2, 0.58532, .05, 1]` = $3.90617

which is the accurate premium. The slack Δ debased from the drift is backed out into volatility $c(\Delta)$ and then replaced again with the skewed volatility measure $c(k, \Delta)\sigma$ negating the effect in the drift for an accurate premium. The skew in volatility backed out from debased prices equated to accurate prices is approximately hyperbolic in the logarithm of the exercise price, as per the approximation and as will be seen with Mathematica graphics.

Skewing volatility in this way (the exact skew) by setting $\sigma^{IV} = c\sigma$ brings the debased premium equated to market prices into parity, by replacing the slack Δ debased from return back into the premium V via the skewed volatility measure $\sigma^{IV} = c(\Delta)\sigma$. Skewing volatility to account for, *by taking up the slack for*, the statistical inaccuracy and bias is not optimal since introducing another slack variable c to compensate for the debasing slack Δ is not minimum variance. The premium is then skewed with slack in both the mean [drift] and the variance [volatility]. Calculating premiums by removing the slack variable Δ directly [from the drift], on the other hand, is the optimal method. The best method, therefore, to go about getting an option price that is notionally accurate and consistent with the market is to re-mean the debased formula [which everyone is familiar with] by using the market's assessment of the future value of the underlying spot market over the deferral period of the derivative by reference to **market put-call parity**. This then adjusts the spot price in the debased option premium to account for the return that has been debased, and implies market return from the market to place it back into the return-free premium [so it is not then return-free]. The volatility measure for the premium is then [straight] **market volatility**, which is estimated for σ with $\hat{\sigma}$ from the data of the underlying spot market. The option market is then used to imply return rather than volatility, where the implied return is an accurate market estimator of market return (Δ) whereas implied volatility is a skewed measure of market volatility ($c\sigma$ to compensate for Δ). Implying return, rather than volatility, is clearly superior but requires that the return of the underlying spot market μ be fully recognized for the time over which the derivative is deferring the spot transaction. This method is based on the assumption that the adjusted spot from the market put-call parity relationship is the market's best estimate of the future return of the underlying spot market, from which it is then implied for the premium. This is also the assumption which allows the volatility smile to be explained.

From the example, remembering that

- `V[10,12,.15,.15,.05,1]` = \$3.90616 is the unbiased premium and
- `V[10,12,.03875,.15,.05,1]` = \$2.5296 is the debased (return-neutral) premium;

instead of skewing volatility, giving the demeaned asset a future value by setting the spot equal to

$Y_t = S_t e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t)} =$ `12 Exp[.15-.5(.15)^2.05[1]]` = \$13.4121

i.e., boosting the spot price of \$12 by 11.77% to \$13.41 to give the notional demeaned asset [for which the debased premium is accurate] a future value with respect to the underlying spot market asset for which the derivative depends. This is the value that is inferred from the option market with **market put-call parity**. Then calculating the debased [return-free] option premium at this adjusted spot rate [replacing the return] results in

`V[10,13.4121,.03875,.15,.05,1]` = \$3.90617

which is the unbiased premium, as required for a sensible and accurate premium. As the suggested method amongst several possibilities, the market provides its estimate on the future value of the asset based on the spot price through the put-call parity relationship; this relationship, given a correct mean, is model-independent since the mean of a distribution is equal to the mean of one side [call option] of that distribution plus the mean of the other side [put option] and only requires that put and call prices correspond to the same model [transition distribution] and assumptions for a given exercise price. For any arbitrary first order diffusion variable and transition distribution of a market asset $S(T)|S(t)$, this relationship is

$$c_{t,T} - p_{t,T} = e^{-r(T-t)}E(S_T \mid S_t) - ke^{-r(T-t)}$$

For proportional Brownian motion, this is

$$c_{t,T} - p_{t,T} = S_t e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t)} - ke^{-r(T-t)}$$

where S_t is the spot price. Therefore, a best estimate for the future value of the market asset is then **implied return inferred from the option market** from **(1)** a call price and **(2)** a put price giving a value for the right hand side $c_{t,T} - p_{t,T} + ke^{-r(T-t)}$ for that required of the left hand side $S_t e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t)}$ of

$$S_t e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t)} = c_{t,T} - p_{t,T} + ke^{-r(T-t)}$$

for Y_t, where *c* is the market call option price and *p* is the market put option price, both for an exercise price *k* expiring at time *T* in $T-t$ periods times, and *r* is the cash rate.

This, $c_{t,T} - p_{t,T} + ke^{-r(T-t)}$, is then the value which is used for the spot value of the demeaned asset *Y* within the debased premium for the demeaned asset, i.e.,

$$Y_t = S_t e^{(\mu+\frac{1}{2}\sigma^2-r)(T-t)} = c_{t,T} - p_{t,T} + ke^{-r(T-t)}$$

for

$$V(k,T;Y_t,t) = Y_t \overline{F}_U\left(u_k - \sigma\sqrt{T-t}\right) - ke^{-r(T-t)}\overline{F}_U\left(u_k\right) \qquad \text{for a call option}$$

$$V(k,T;Y_t,t) = ke^{-r(T-t)}F_U\left(u_k\right) - Y_t F_U\left(u_k - \sigma\sqrt{T-t}\right) \qquad \text{for a put option}$$

with $u_k = \dfrac{\log k - \log Y_t - (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$

which remeans the premium. For the example, \$13.4121 would be obtained from the sum of **(1)** the difference between a market call and market put [for a given exercise price *k*] and **(2)** the discounted exercise price

$$c_{t,T} - p_{t,T} + 10e^{-0.05(1)} = \$13.4121$$

so that the market would have provided the value of the difference between the put and the call option of

$$c_{t,T} - p_{t,T} = \$13.4121 - 10e^{-0.05(1)} = \$3.8998$$

This will account for both put-call parity and structurally skewed volatility [with respect to *k*] under the assumption that the option market **(1)** implicitly takes into account asset returns of the spot markets and **(2)** has an aversion away from flatlining itself and each and every one of its option payoff profiles. Note that the market return, net of cash, inferred from the option market is then

$$\mu + \tfrac{1}{2}\sigma^2 - r = \frac{1}{T-t}\log\frac{\hat{C} - \hat{P} + ke^{-r(T-t)}}{S_t}$$

From the example, this is then $\mu + \frac{1}{2}\sigma^2 - r = \frac{1}{1}\log\frac{13.4121}{12}$ = 11.13% pa implied from the market[10].

This technique, based upon market put-call parity, can then be used as a **crystal ball** for returns of the spot markets; its goodness? that is yet to be seen, but this thesis bets that those who know what they are doing and use it intelligently will get real predictive power, as well as insight into the future [in the spot markets by reference to the derivative markets] and insight into the personality that is the market; a bull or bear, for example, with respect to returns [with a numerical value]. This will also allow for ex-post empirical studies on the **crystal ball** regarding how well the derivative markets, for deferred spot transactions, does in its assessment of future risk and return of the spot markets [via the option markets], and how this relates to issues regarding the interaction between **(1)** market sentiment, **(2)** supply and demand of securities, and **(3)** the bid and the ask in placing prices on listed and traded financial securities for which vary stochastically as variables on *S*, i.e., all of the determinants of the **crystal ball** that is the derivative markets.

The only item that then remains is volatility itself, which is notionally market volatility and is hence estimated by maximum likelihood from the data of the spot market. It can also be inferred from remeaned prices, which are then notionally free from Δ, and therefore should be free from systematic skew over exercise price *k* since it is then free from the slack Δ that generates the skew *c*(Δ,*k*) in volatility with respect to *k*. Hence, implying volatility after returning the premium to the statistically accurate premium then adjusts for any issues regarding the markets assessment of upside and downside risk and should account fully for the structural deviations in σ with respect to the exercise price *k*.. If volatility is inferred after the premium is remeaned with **market put-call parity**, the implied measure should to be consistent with market volatility. Noting that inferring Δ from the market is still just an estimate, so notional sampling variation will always be present [although it should not be system-

[10] which is .15+0.5×.15^2–.05 from the parameters of the example.

atic with k] in the estimation of Δ in order to remove it from the debased premium. A discrepancy between implied volatility for upside (calls) or downside (puts) can, within reasonableness, be attributed to certainty or uncertainty in upside (for calls) or downside (for puts) [it could also be a remnant to volatility being nudged differently for puts and calls to vary [tweak] debased prices {in a suboptimal attempt to remove Δ to get a more accurate price}]. This could well prove to be another element to the **crystal ball**. Further, a derivatives market allowed to find its level with regard to return, and free from debasing slack in prices, will tend towards a more natural market equilibrium which is more consistent with **efficient and free markets**, and the **unique characteristics of derivatives**. For example, calls and puts more accurately priced, after remeaning debased prices, will make puts more marketable so that the ratio of the number of calls to number of puts will be a better measure of market sentiment (bullish or bearish), and the circumstance of ordinarily less puts in the market will be for the right reasons, namely, that it is the derivatives equivalent to being a short trader [which comprise a smaller fraction of the market makers], rather than puts being systematically overpriced and hence less marketable because of a mispricing in the premium [and instead because it is a position that is against the growth pattern of the market].

Matching **market put-call parity** to infer return for option prices is not perfect but it is superior to any alternative. It is also a circularity; *using market prices to get a market price* [although not a pointless circularity]. So from time-to-time, it should be referenced against other methods including CAPM and, most importantly, purely using data with maximum likelihood from the underlying spot market [which is the true reality check]. It can also be referenced with the futures market by the same principles [looking at the exercise prices of zero premium futures].

A more sophisticated premium may be chosen by reference to the stochastic calculus from BOOK 3 of this PhD thesis for which a proportional Brownian motion is but one simple possibility. Matching **market put-call parity** to infer return is also available [possible] with any of these variables and the resulting premiums, since put-call parity is a mathematical relation that applies to any distribution which is defined over a sensible sample space. All it requires is a transition mean. The put-call parity relationship is then

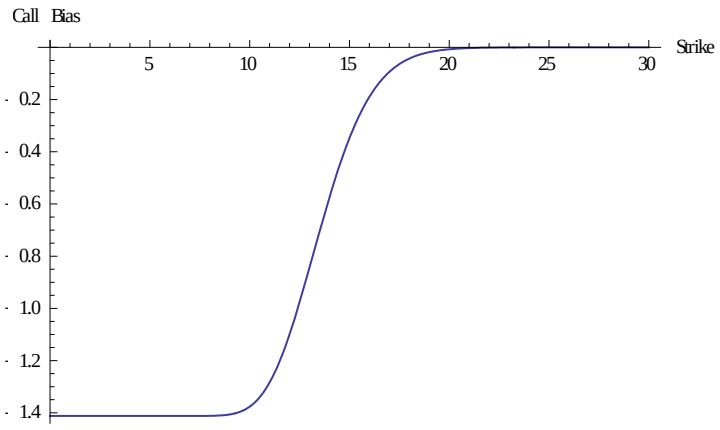
$$c_{t,T} - p_{t,T} = e^{-r(T-t)} E(S_T \mid S_t, DS_t, D^2 S_t, ...) - k e^{-r(T-t)} \qquad \text{where} \quad D = \frac{d}{dt}$$

for higher order diffusion variables with the **extended Markov property**[11] considered in BOOK 3 .

Numerical Example: Call Bias and Skewed Volatility for Variable Exercise Price

The statistical bias of the debased call option premium for a variable exercise price;

```
Plot[ V[ x, 12, .03875, .15, .05, 1] - V[ x, 12, .15, .15, .05, 1] , { x, 0, 30} ,
 AxesLabel → { "Strike", "Call Bias"} ]
```

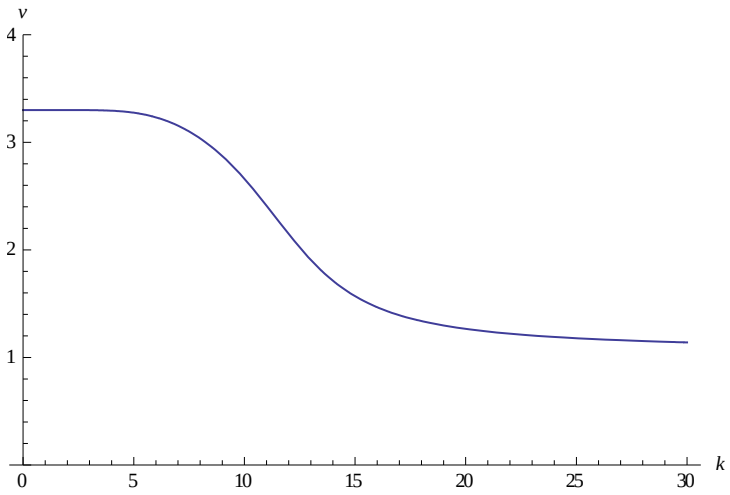


which shows the functional shape of the statistical bias and inaccuracy of the debased call option as a function of the exercise price k ; this curve is something similar to a logistic function and is greatest in magnitude when calls are inthemoney and negligible when calls are outofthemoney. The call approaches zero as it becomes more outofthemoney, since there is little or no probability that it will expire inthemoney and the bias approaches zero as both the accurate and debased premiums approach zero.

[11] higher order dependence on its past.

- Skewing volatility in the second cumulant only

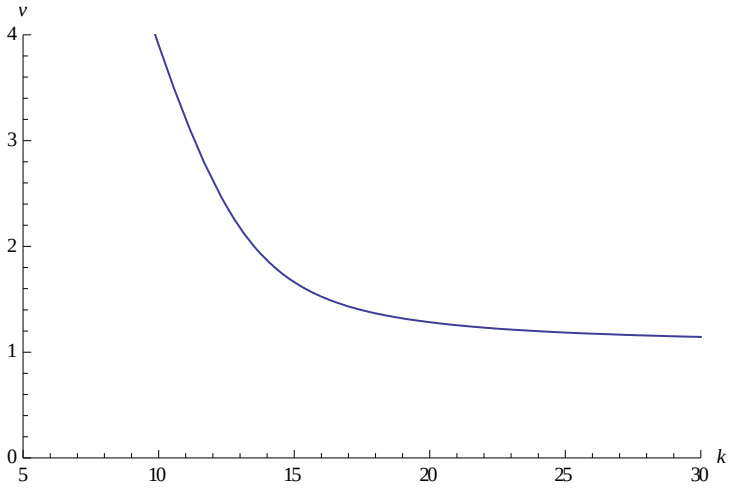
```
Plot[ FindRoot[ V[ k, 12, .15, .15, .05, 1] - V[ k, 12, .05 + .01125, .15 v, .05, 1] == 0,
    { v, 1.1}]][[ 1, 2]] ,{ k, 0, 30} , PlotRange → { 0, 4} , AxesLabel → { k, v} ]
```



which is a smooth continuous function, unlike that which also skews volatility present in the mean $\frac{1}{2}\sigma^2$.

- Skewing volatility in the second cumulant and within $\frac{1}{2}\sigma^2$ in the first cumulant, which is the exact volatility skew implied from the debased return-neutral call option premium

```
Plot[ FindRoot[ V[ k, 12, .15, .15, .05, 1] - V[ k, 12, .05 + .01125 v^2, .15 v, .05, 1] == 0,
    { v, 1.1}]][[ 1, 2]] ,{ k, 2, 30} , PlotRange → {{ 5, 30} ,{ 0, 4} } , AxesLabel → { k, v} ]
```



This is the exact skew arising from the statistical bias in the debased call option formula. It is a nice smooth smirk.

- How does this compare to what has been observed in the literature? Here is an empirical volatility smile from the Journal of Finance (1998)[12]

[12] Dumas, B., Fleming, J. and Whaley, R. E., (1998) Implied Volatility Functions: Empirical Tests. *The Journal of Finance*, Vol. 53, No. 6 (Dec., 1998), pp. 2059-2106.

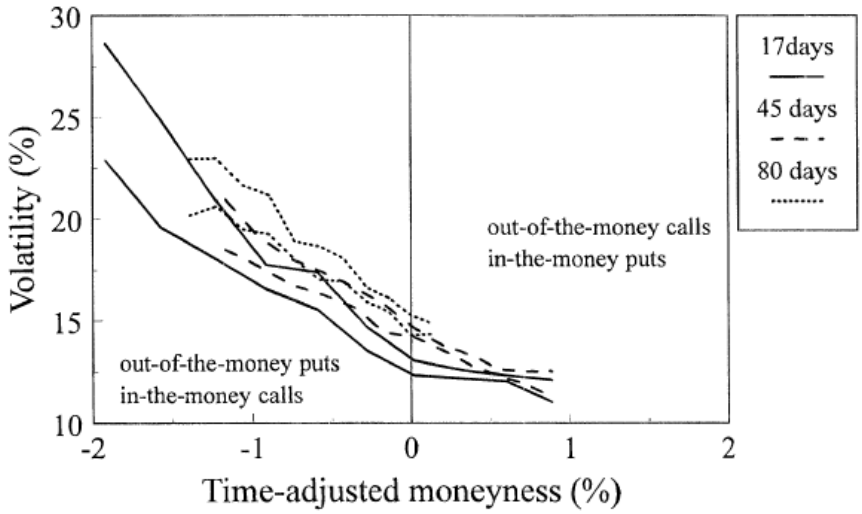


Figure 1. Black–Scholes implied volatilities on April 1, 1992. Implied volatilities are com-

The explanation for systematic volatility skew is simple; it is volatility taking up the slack for the bias in the debased formulae equated to market prices. Other **excerpts** from this paper:

Strikingly, the volatilities do not all lie on a horizontal line. This pattern is often called the volatility "smile" and constitutes evidence against the Black–Scholes model. In the figure, the "smile" actually appears to be more of a "sneer."

and

VIII. Summary and Conclusions

Claims that the Black and Scholes (1973) valuation formula no longer holds in financial markets are appearing with increasing frequency. When the Black–Scholes formula is used to imply volatilities from reported option prices, the volatility estimates vary systematically across exercise prices and times to expiration.

It never held for financial markets, as per Black and Scholes (1973) and their admission that the (their?) debased call option formula varies systematically with the exercise price and is biased downwards against market prices [and the opposite for their put option formula if they had have also looked at put option data for their 1973 journal article]. If the debased formula is theirs, the accurate formula is mine. Below is an excerpt from the 1973 Black and Scholes[13] paper *The Pricing of Options and Corporate Liabilities*

[13] Black, F. and Scholes, M. J., (1973) The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, Vol. 81, No. 3 (May-Jun., 1973), pp. 637-659.

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Empirical Tests

We have done empirical tests of the valuation formula on a large body of call-option data (Black and Scholes 1972). These tests indicate that the actual prices at which options are bought and sold deviate in certain systematic ways from the values predicted by the formula. Option buyers pay prices that are consistently higher than those predicted by the formula. Option writers, however, receive prices that are at about the level predicted by the formula. There are large transaction costs in the option market, all of which are effectively paid by option buyers.

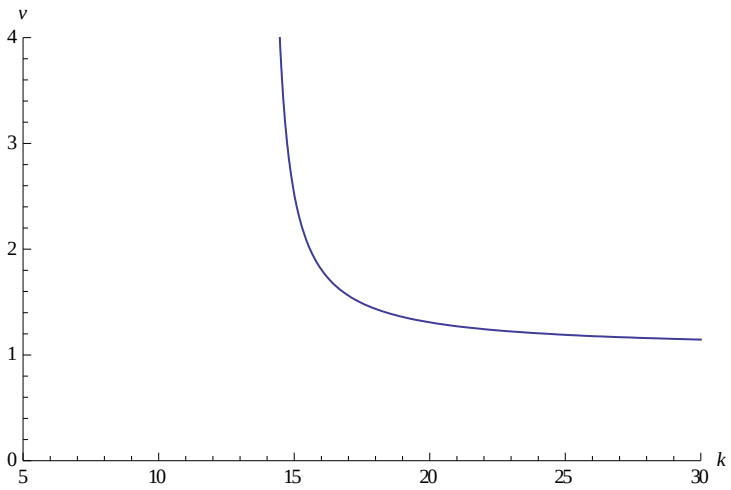
admission by Black and Scholes that the call option market price is higher than the debased price; this is just evidencing the inaccuracy of the debased price.

The authors missed the implication of their observation that market prices of call options are above those suggested by their systematically deviant formula for a call option premium [and have tried to explain it by transaction costs {compare the word ***consistently*** with ***about***}; note that there is only one transaction price for any trade]. This empirical observation is consistent with markets whereby market assets earn more than a bank account, on average, and the option market implicitly pricing more correctly than the debased premium which is accurate only for risky market assets that are void of return, i.e., empirically evidencing only that derivative markets do not kill return μ in prices with slack Δ. *Such premiums are definitely not consistent with other financial or economic theory.*

Put options are not mentioned but the equivalent empirical finding for put options would be that market prices for puts are consistently lower than that suggested by the formula for the debased price of a put option, which by their own admission for calls, is systematically deviant, i.e., systematically misprices options. The systematic mispricing in the debased price is therefore the determinant of the systematic curve in implied volatility as a function of exercise price.

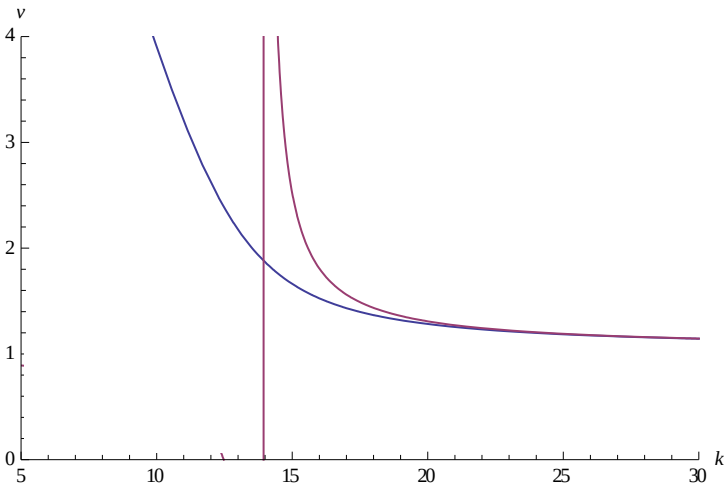
- The simple analytic approximation $f(k)$ for the volatility skew $c(k)$ for out of the money options

```
f[k_] := 1 + .11125 (Log[k] - Log[12]) - .15
Plot[f[k], {k, 14, 30}, PlotRange -> {{5, 30}, {0, 4}}, AxesLabel -> {k, v}]
```



- How does the approximation compare with the exact skew?

```
Plot[
 {FindRoot[V[k, 12, .15, .15, .05, 1] - V[k, 12, .05 + .01125 v^2, .15 v, .05, 1] == 0,
   {v, 1.1}][[1, 2]], f[k]}, {k, 2, 30}, PlotRange -> {{5, 30}, {0, 4}}, AxesLabel -> {k, v}]
```



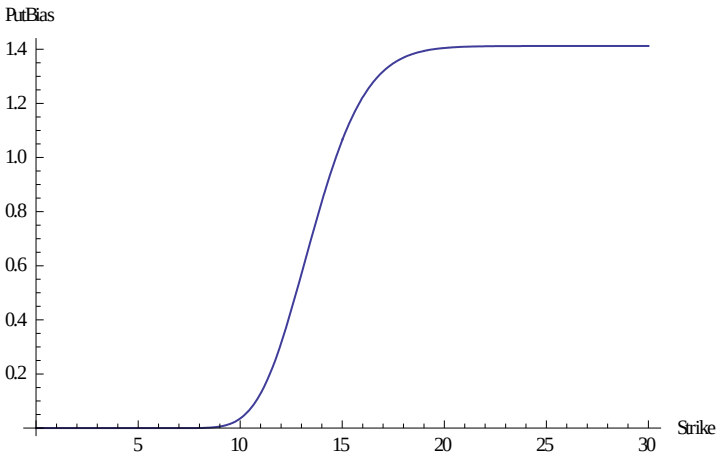
not bad for the simple analytic form over the applicable range, which is to the right of the pole, $k > \exp(2.6349) = 13.94$.

Numerical Example: Put Bias and Skewed Volatility for Variable Exercise Price

```
P[k_, S_, m_, s_, r_, t_] :=
  k Exp[-r t] CDF[NormalDistribution[0, 1], (Log[k / S] - m t) / Sqrt[s^2 t]] -
   S Exp[(m - 0.5 s^2 - r) t] CDF[NormalDistribution[0, 1],
     (Log[k / S] - m t) / Sqrt[s^2 t] - Sqrt[s^2 t]];
```

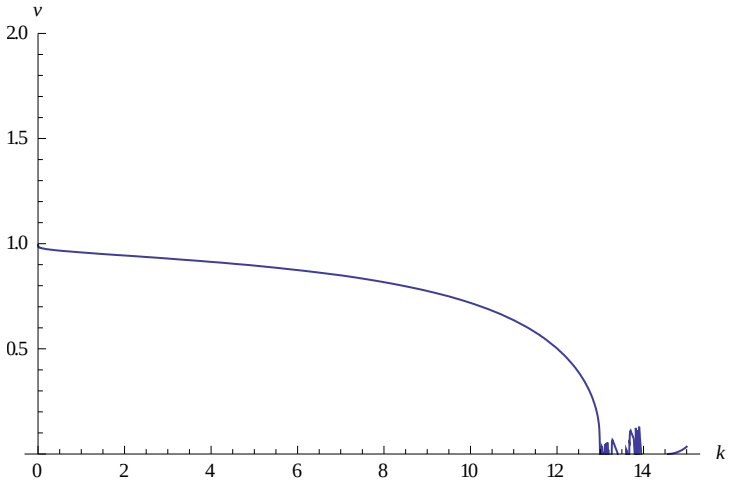
- put option bias

```
Plot[P[x, 12, .03875, .15, .05, 1] - P[x, 12, .15, .15, .05, 1], {x, 0, 30},
 AxesLabel -> {"Strike", "Put Bias"}]
```



- skewed volatility for a put option; volatility has to be nudged downwards to reduce the overpriced de-based put option premium

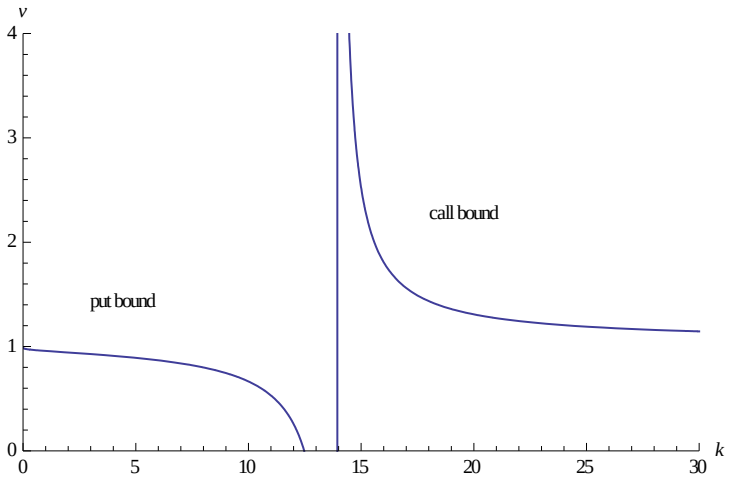
```
Plot[
 FindRoot[P[k, 12, .15, .15, .05, 1] - P[k, 12, .05 - .01125 v^2, .15 v, .05, 1] :=
    0, {v, 1}][[1, 2]], {k, 0, 15}, PlotRange -> {0, 2}, AxesLabel -> {k, v}]
```



plus lots of crappy unpleasantness and poles above the point at which the skew *c* first zeroes, which is nothing other than an artefact of the root finder. Volatility is squared in the premium formulae, so as it decreases for a put it cannot become negative and it cannot be adjusted further downwards past this point even although the bias is still increasing with *k*. Past this point, the solutions for the skew *c* are not defined and are not sensible.

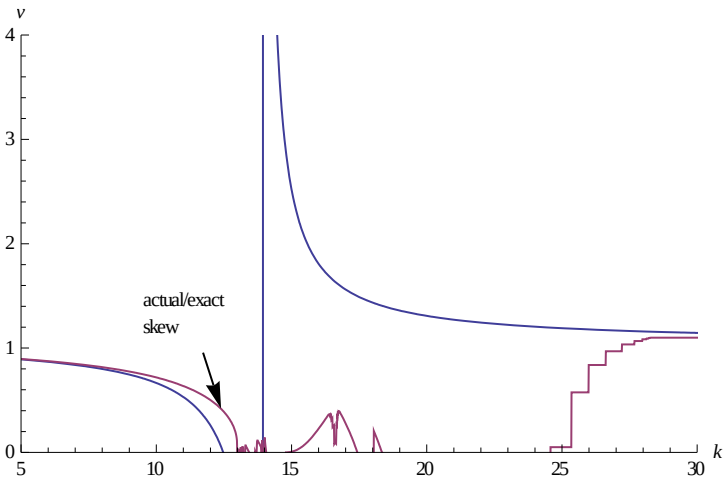
- formula for analytical approximation

```
f[k_] := 1 - .11125 (Log[k] - Log[12]) - .15
Plot[f[k], {k, 0, 30}, PlotRange -> {{0, 30}, {0, 4}}, AxesLabel -> {k, v}]
```



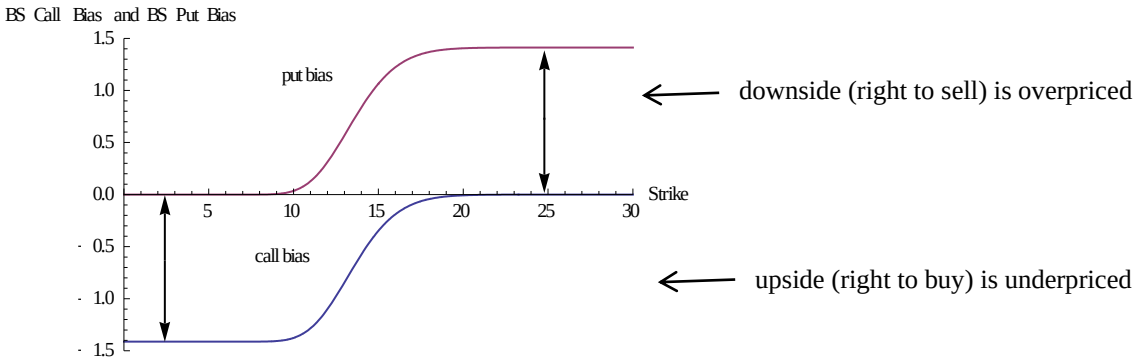
- how does it compare?

```
Plot[
 {f[k], FindRoot[P[k, 12, .15, .15, .05, 1] - P[k, 12, .05 - .01125 v^2, .15 v, .05, 1] == 0,
    {v, 1}][[1, 2]]}, {k, 0, 30}, PlotRange -> {{5, 30}, {0, 4}}, AxesLabel -> {k, v}]
```



very well before the asymptote, which is the range for which the approximation is defined for a put option; thereafter there is unpleasantness of inconsistency, numerical error, and convergence issues for the exact skew and the root finder. Why? because a put option is overpriced by a debased return-free option premium, and the greater the exercise price, the greater the statistical bias and overpricing, and the smaller volatility must be made to compensate; but skewed volatility has a root [and sign change] just before the exercise price. Volatility cannot be negative, so after this root, skewed volatility is not defined for real numbers [before this for calls]. Skewed volatility is not defined for low exercise price calls and high exercise price puts, i.e., for very inthemoney options. This invalidates skewing volatility to get an accurate and statistically unbiased premium as a general method to get an option price. Hence, backing out skewed volatility is definitely not the sensible way to go about compensating for a debased premium.

- call and put bias for debased return-neutral option premiums, $C^{debased} - C$ and $P^{debased} - P$, compared:



which is pretty obviously the generator of the skew in volatility.

Conclusion

Skewing volatility is not an optimal method for obtaining an accurate option premium consistent with the market.

PART II: SUMMARY AND CONCLUSION

This thesis stresses the notion of trying to be accurate, in an uncertain world, and accurate for stochastic variables on *S*. Free from the debased nonsense, which **impairs any attempt at being rationale about managing risk and diffusion variables on *S***, the stochastic calculus of diffusion variables of arbitrary order of BOOK 3 is then available to model market variables with more sophistication; as an integral and differential rate of change calculus for stochastic basis functions on *S* which, it is asserted, compose both the spot and deferred financial markets. This is in great contrast to a one size fits all formula that systematically misprices all options. This is has been the main purpose for writing BOOK 2; for BOOK 3 to be applied to options within the context of striving to be accurate in an uncertain world for a better, more honest and professional financial system, and a better

world for all (less insane things happening to people {like people getting gang raped in banks[14]})]. The mathematics is no longer a 'one-size-fits-all' and is not simply a case of plugging numbers into a restricted set of formulas but a calculus, as an extension to Newton, as a methodology that requires sound mathematical and statistical intuition and judgment.

Every time that someone has observed and acknowledged that **(1) market prices do not equate with prices suggested by the debased premium, (2) that implied volatility varies systematically over variations in the exercise price for the debased premium, or that (3) debased put-call parity of the debased premium does not hold for market prices**, they have been observing and acknowledging that return-neutral valuation is outright wrong and that option prices are not debased prices void of market return, i.e., that market prices for derivatives do not kill return μ with slack Δ. The technique of this thesis for correcting the debased option price [the return-free price (killed by Δ)] by adjusting the spot by reference to market put-call parity [to remean the debased price (by removing Δ)] accounts and corrects for both these issues, and is superior to skewing volatility to get an accurate price since it is optimal in that it is minimum variance [unlike skewing volatility].

The return-neutral methodology states that every option must be return-neutral [debased with Δ] and that every option market position must be hedged [replicated] and flatlined by trading with an offsetting position which implies that there are no remaining option exposures while there are a lot of very busy people doing a whole lot of trading in order to subsidize each other's option premiums. Every premium in such a market then each represents a null claim from a flatlined payoff profile; and if it worked, which it doesn't, for nothing other than a flatlined and redundant market. The conceptual positioning of the return-neutral methodology is pointless and not an efficient market philosophy for the pricing and trading of option derivatives, and not a sufficient reason to systematically misprice derivatives. Apart from the fact of being inefficient, the return-neutral methodology defeats the purpose for the very existence of derivatives.

The panacea for risk that is suggested by risk-neutral valuation is at best misleading and the closest thing that there is to a panacea for risk in regard to derivatives is **(1)** professional statistical judgment and intuition and **(2)** knowledge and understanding of the fundamentals of markets. This is especially so in the context, here, of the efficient functioning of the exchanges, finance, and large sums of corporate funds that have bearing and influence on national and global economies, and why it is so important that the return-neutral nonsense changes. Most noteworthy for return-neutral valuation, to desist in discrediting sensible people with acts of dishonesty and remove the need to turn to falsities to make up for inadequacies in theoretical positioning; by getting it wrong every single time(!) and then having to lie and discredit people in order to try and protect oneself. The finances and ownership of some of the largest corporations in the world trade on such models [and the majority of them on the exchanges], it is vitality important the models on which these securities trade are accurate and not based upon **demeaned nonsense**, and **charlatan delusions of having no risk**. It is no wonder that, in important circumstances, the numbers have stopped 'adding up' and that corporate insolvencies and scandals were not identified earlier and stopped. It is absolutely clear why a profession (and Hippocratic oath with regard to making honest statements in a professional capacity), one with moral, ethical, and fiduciary standards and that which has the ability to regulate its members (and **punish** where appropriate and/or **expel** [with a Disciplinary Scheme?!]), and ensure that its members act in a proper and technically competent manner free from **negligence**, **incompetence**, and **wrongdoing**[15], is put in place to manage the risk aspects and regulation of option related contingencies based on sensible and technically correct mathematics. Return-neutral aspects of finance where never fully understood and where never in touch with the reality of markets, and have caused finance to lose touch with the rest of its science and engineering sibling disciplines. The calculus of this thesis, with the moral and ethical standards and technical competencies of a well regulated and ethical profession, will bring finance back in touch with reality. It will also bring the technical people back in touch with the options people and a profession as a baseline to ensure that everything is functioning as it should be. The calculus of this thesis will work best, as well as contribute to efficient, competitive, and well functioning financial markets given those with the ability [mathematical high achievers[16]] to make the most of the calculus from BOOK 3 of this thesis.

A professional apparatus is already in place for the risk management of derivatives; an **examination process** and **technical standards** and **people** of the actuarial profession. All it requires for this step, as a new practice area, is to recognize that the debased nonsense is wrong, prices need to be accurate, and the diffusion calculus of arbitrary order of BOOK 3 of this thesis is ready for the management of derivatives as securities which derive from integral and differential dynamics on *S*. With the stochastic calculus of BOOK 3, in the context of accurate premiums and pricing, the scope for a new actuarial practice area is unequivocal.

[14] Macquarie Bank to be more specific.
[15] most particularly, free from exploitation of positions of trust and responsibility.
[16] and not hopeless inept return-neutral communists that do not know the importance [or the definition] of statistical accuracy [and fraudulently discredit and fail those who do].